Minimal renormalization without ε expansion: Four-loop free energy in three dimensions for general n above and below T_c

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We present an analytic four-loop calculation of the free energy in three dimensions within the $O(n)$ symmetric φ^4 theory at infinite cutoff for general n above and below T_c. It is shown that Goldstone singularities arising at intermediate stages of the calculation cancel among themselves. The correlation length above T_c and an appropriately defined pseudocorrelation length below T_c are calculated analytically up to four-loop order for general n. The method of minimal renormalization at fixed dimension $d=3$ is used to determine the analytic expressions for the four-loop series of the amplitude functions of the free energy, correlation length, and specific heat above and below T_c in terms of the renormalized coupling. These expressions provide the basis for future accurate Borel resummations of universal amplitude ratios characterizing the asymptotic critical behavior and of crossover functions describing the nonasymptotic critical behavior. A brief application is given by a variational calculation of the universal specific-heat amplitude ratios A^+/A^- and $P=\alpha^{-1}(1-A^+/A^-)$ and of the universal quantity $R_\xi^+=\xi_0^+(A^+)^{1/d}$ for general n.

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I. INTRODUCTION

One of the important achievements of the renormalization-group (RG) theory is the identification of universality classes for the critical behavior of thermodynamic systems [1,2]. Critical exponents, ratios of asymptotic amplitudes, and scaling functions of thermodynamic quantities are predicted to depend only on the dimension d of the system and the number n of components of the order parameter.

This unifying feature of bulk universality holds not only for ideal systems with purely short-range interactions but also for real systems with subleading long-range interactions, such as fluids with van der Waals interactions. Universality is not generally valid, however, for the sizedependent part of the scaling functions of confined fluids [3]. Even for systems with purely short-range interactions, universality is not valid for the exponential large-distance behavior of the bulk order-parameter correlation function in the asymptotic critical region above T_c [4].

Of fundamental interest is the critical behavior of the free energy density f and the specific heat C. The goal of the present paper is to present field-theoretical results for the amplitudes of these quantities above and below T_c for general n in $d=3$ dimensions, which provide the basis for improving the accuracy of the theoretical predictions for measurable universal amplitude ratios and crossover functions.

We consider the Gibbs free energy $f^\pm(t)$ per unit volume, divided by k_BT, at zero-ordering field and at the reduced temperature $t=(T-T_c)/T_c$ above $(+)$ and below $(-)$ T_c. It is expected that, for small $|t|$, the free energy can be decomposed as [5]

$$f^\pm(t)=f_s^\pm(t)+f_{ns}(t), \tag{1}$$

with the "singular part"

$$f_s^\pm(t)=-\frac{A^\pm}{\alpha(1-\alpha)(2-\alpha)}|t|^{2-\alpha}[1+a_f^\pm|t|^\Delta+\cdots] \tag{2}$$

and the "nonsingular part"

$$f_{ns}(t)=f_0+f_1t-\tfrac{1}{2}Bt^2+O(t^3). \tag{3}$$

In Eq. (2) we have included the leading Wegner corrections to scaling [6]. Near T_c, the leading critical behavior of the specific heat per unit volume, divided by k_B, is obtained from $f^\pm(t)$ as

$$C^\pm(t)=-\frac{\partial^2f^\pm}{\partial t^2}=C_s^\pm(t)+C_{ns}(t), \tag{4}$$

where

$$C_s^\pm(t)=\frac{A^\pm}{\alpha}|t|^{-\alpha}[1+a_c^\pm|t|^\Delta+\cdots], \tag{5}$$

$$C_{ns}(t)=B+O(t). \tag{6}$$

The specific heat is the most favorable candidate for a quantitative test of the predictions of the RG theory. These predictions include the universality of the ratios A^+/A^- and $a_c^+/a_c^-=a_f^+/a_f^-$ [5,7]. This is of particular interest for the $d=3$, $n=2$ universality class, where detailed experimental tests of universality have been performed in the past [8] and where new high-precision estimates are available from recent experiments in space [9,10] and from numerical investigations [11]. Additional experimental tests of universality along the λ line of ^{4}He are planned for future research [12].

Equations (3) and (6) imply that, for negative α, the specific heat is *continuous* at T_c with a finite nonuniversal value,

$$\lim_{t\to0+}C^+(t)=\lim_{t\to0-}C^-(t)=C_{ns}(0)=B>0. \tag{7}$$

To the best of our knowledge, however, there exists no general proof in the literature on the continuity of $C^{\pm}(t)$ for $\alpha<0$. On the contrary, on the basis of a field-theoretic RG analysis the possibility has been suggested [13] that, for $\alpha<0$, $\lim_{t\to 0+}C^{+}(t)=B^{+}\neq B^{-}=\lim_{t\to 0-}C^{-}(t)$ corresponding to a discontinuity of the specific heat at T_c.

The nonuniversal amplitudes $(A^{\pm})^{1/d}$ have the dimension of inverse lengths. Above T_c, an additional length scale is provided by the (second-moment) correlation length

$$\xi_{+}=\xi_0^{+}t^{-\nu}. \tag{8}$$

In accord with the hypothesis of two-scale factor universality or "hyperuniversality" [14], however, RG theory [7,15] predicts a universal relation between the lengths ξ_0^{+} and $(A^{+})^{-1/d}$ such that

$$R_{\xi}^{+}=\xi_0^{+}(A^{+})^{1/d} \tag{9}$$

is a universal quantity. Together with the hyperscaling relation [1,2]

$$d\nu=2-\alpha, \tag{10}$$

this implies that the singular part of the free energy in the correlation volume $(\xi_{+})^{d}$

$$\lim_{t\to 0}f_s^{+}(t)(\xi_{+})^{d}=-\frac{(R_{\xi}^{+})^{d}}{\alpha(1-\alpha)(2-\alpha)}, \tag{11}$$

is universal. There exists also a corresponding universal ratio below T_c [5,7].

In the field-theoretic RG calculations of universal amplitude ratios, two different approaches have been used, (i) the $\varepsilon=4-d$ expansion [16], (ii) calculations at fixed dimension $d=3$ [17]. Within these two approaches one can further distinguish between two types of renormalization: (a) the use of renormalization conditions [18] and (b) the minimal subtraction scheme [19]. Part of the results for universal amplitude ratios have been obtained within the ε expansion [20,21] and within the $d=3$ approach using renormalization conditions [22–28]. As an alternative approach, a combination of the minimal subtraction scheme with the $d=3$ approach, without an ε expansion, was proposed [29–31] and applied to various universal amplitude ratios and amplitude functions [29–37].

While the field-theoretic calculations have been performed for critical exponents partially up to seven-loop order [28,38,39] and for amplitudes *above* T_c up to five-loop [27,32,35], six-loop [22] and partially seven-loop [28,40] order, considerably less is known for amplitudes *below* T_c. In this case, five-loop [23,27,33] and partially seven-loop [28,40] results are available only for $n=1$ on the basis of the five-loop expression for the $n=1$ free energy [23]. For the important case $n\geqslant 2$ there exist field-theoretic results only up to two-loop [20,21,31,34] and three-loop [36,37] order. This is due to the fact that analytic calculations below T_c for $n>1$ are significantly more difficult than for $n=1$ because of the existence of both longitudinal and transverse fluctuations of the order parameter. These difficulties are related to spurious Goldstone divergences at intermediate stages of the perturbation theory unless an external ordering field is kept finite until the end of the calculations [34,36]. Even for $n=1$ the amplitude of the correlation length below T_c requires nontrivial calculations at finite wave vector, which have so far been carried out only up to three-loop order [25,26].

In the present paper we perform the next substantial step by deriving the analytic perturbative expressions for the Gibbs free energy $f^{-}(t)$ and the specific heat $C^{-}(t)$ below T_c up to *four-loop* order for general *n* within the $O(n)$ symmetric φ^4 theory. The starting point is the perturbative four-loop expression of the bare Helmholtz free energy derived recently [41]. The desired information is the four-loop contribution to the amplitude functions of f^{-} and C^{-} in terms of the renormalized coupling. We employ the method of combining [29,30] the minimal subtraction scheme with the massive field theory at fixed dimension $d=3$, involving an appropriately defined pseudocorrelation length ξ_{-} [31]. Our approach has the advantage of being applicable both above and below T_c with the same multiplicative and additive renormalizations, unlike $d=3$ RG theories using renormalization conditions. At the two-, three-, and four-loop level, this implies substantial simplifications in the analytic calculation of universal combinations of amplitude functions above and below T_c.

As a by-product we also obtain *analytic* perturbative expressions for $f^{+}(t)$, $\xi_{+}(t)$, and R_{ξ}^{+} at $d=3$ for general *n* up to four-loop order, whereas the previous higher-loop results [22,27,28,32,35,40] above T_c were restricted to $n=1,2,3$ in numerical form. Our results above T_c provide the basis for predicting R_{ξ}^{+} for arbitrary *n* including the approach to the exactly known [5,42] limit $n\to\infty$.

Furthermore, we take up the question regarding the continuity of $C^{\pm}$ at T_c for $\alpha<0$. We show that $C^{\pm}$ is continuous at T_c for $\alpha<0$ within the φ^4 theory at infinite cutoff in $2<d<4$ dimensions. This is at variance with Fig. 3(b) of Ref. [13]. We note, however, that a more complete proof is required involving a RG treatment at *finite* cutoff, since the critical value *B* of $C^{\pm}$ is nonuniversal and depends on the cutoff procedure.

Our analytic four-loop results for $f^{\pm}(t)$, $\xi_{\pm}(t)$, and $C^{\pm}(t)$ are not restricted to the asymptotic critical region but contain sufficient information to derive the nonasymptotic contributions of the Wegner expansion (at infinite cutoff). This information can also be used for quantitative studies of nonasymptotic critical behavior and of the crossover between critical and noncritical behavior by means of a nonlinear RG analysis [43,44].

Since the loop expansions of the φ^4 theory are not convergent it is necessary to use resummation techniques. For recent reviews see Refs. [45,46]. Two methods are available: (i) Borel resummations [22,23,27,28,32,47] and (ii) order-dependent mapping and variational approach [48–55]. An application of these methods to our four-loop results including a careful determination of error bars is the next important step, which is beyond the scope of the present work and will partially be performed in a separate paper [47].

Here we confine ourselves to a brief application of the variational approach to estimate $P=\alpha^{-1}(1-A^+/A^-)$ and R_ξ^+ for general n, without an estimate of error bars. Our variational four-loop results are close to the experimental [9,10] and numerical results [11] and Borel resummations [22,23,27,32,33,35–37,47]. The Borel-resummed result of P for $n=2$ of the most recent work [47] is based on the four-loop series of the present paper and is more accurate than the most recent numerical estimate for P within the three-dimensional XY model [11], the high-precision experimental result for P near the superfluid transition of ^{4}He [9,10], and the Borel-resummed values based on earlier three-loop series [35,37]. Our four-loop variational estimates also improve the corresponding variational estimates [55] based on our earlier three-loop results [36].

The outline of our paper is as follows. In Sec. II we present the analytic four-loop expression of the bare Helmholtz free energy of the $O(n)$ symmetric φ^4 theory for general n in three dimensions near the coexistence curve below T_c. Section III serves to provide the analytic perturbative relations between the bare correlation lengths and the temperature variable above and below T_c up to four-loop order. The bare Gibbs free energy in the limit of vanishing ordering field is calculated up to four-loop order in Sec. IV. The renormalized version of the theory is presented in Sec. V, where the power series of the amplitude functions of the specific heat and the correlation lengths are given up to four-loop order. In Sec. VI the results of Sec. V are applied to the asymptotic critical region. A variational calculation of universal amplitude ratios for general n in three dimensions is presented in Sec. VII. The Appendixes contain important complementary information.

II. BARE HELMHOLTZ FREE ENERGY

We consider the standard Landau-Ginzburg-Wilson functional

$$\mathcal{H}=\int_V d^d x\left[\frac{1}{2}r_0\varphi_0^2+\frac{1}{2}\sum_i(\nabla\varphi_{0i})^2+u_0(\varphi_0^2)^2-\mathbf{h}_0\cdot\boldsymbol{\varphi}_0\right], \tag{12}$$

$$r_0=r_{0c}+a_0 t,\quad t=(T-T_c)/T_c \tag{13}$$

for a d-dimensional system of volume V with an n-component field $\boldsymbol{\varphi}_0(\mathbf{x})=(\varphi_{01}(\mathbf{x}),\ldots,\varphi_{0n}(\mathbf{x}))$ in the presence of the homogeneous external field $\mathbf{h}_0=(h_0,0,\ldots,0)$. The spatial fluctuations of $\boldsymbol{\varphi}_0(\mathbf{x})$ are restricted to wave numbers less than a cutoff Λ. A factor $1/k_BT$ is absorbed in $\mathcal{H}$. The Gibbs free energy per unit volume (divided by k_BT) is

$$F_0(r_0,u_0,h_0,\Lambda)=-V^{-1}\ln\int\mathcal{D}\boldsymbol{\varphi}_0\exp(-\mathcal{H}). \tag{14}$$

For a comparison with experiments, a "background" Hamiltonian $\mathcal{H}_B(T)$ must be added to Eq. (12), which describes the effect of degrees of freedom other than $\varphi_0(\mathbf{x})$ and contributes an additive regular part

$$f_B=\mathcal{H}_B/V=f_0^B+f_1^B t-\tfrac{1}{2}C_B t^2+O(t^3) \tag{15}$$

to F_0. Additional fluctuation-induced regular terms are contained in F_0 as will be discussed below. We shall always consider the bulk limit $V\to\infty$. The Helmholtz free energy per unit volume Γ_0 is obtained from F_0 via the Legendre transformation

$$\Gamma_0(r_0,u_0,M_0,\Lambda)=F_0(r_0,u_0,h_0,\Lambda)+h_0M_0, \tag{16}$$

where $M_0(r_0,u_0,h_0,\Lambda)=\langle\varphi_{01}\rangle=-\partial F_0/\partial h_0$ is the order parameter (magnetization). The perturbative expression of Γ_0 is given by the mean-field term minus the sum of the one-particle irreducible vacuum diagrams. The structure of the analytic expression is (apart from an unimportant additive constant that can be absorbed into f_0^B)

$$\begin{aligned}\Gamma_0(r_0,u_0,M_0,\Lambda)&=\frac{1}{2}r_0M_0^2+u_0M_0^4+\frac{1}{2}\int_{\mathbf{p}}^{\Lambda}\ln(\overline{r_{0L}}+p^2)\\&\quad+\frac{1}{2}(n-1)\int_{\mathbf{p}}^{\Lambda}\ln(\overline{r_{0T}}+p^2)\\&\quad+\sum_{b=2}^{4}u_0^{b-1}X_0^{(b)}(r_0,u_0,M_0,\Lambda)+O(u_0^4),\end{aligned} \tag{17}$$

where $\int_{\mathbf{p}}^{\Lambda}\equiv(2\pi)^{-d}\int^{\Lambda}d^dp$ means integration up to $|\mathbf{p}|=\Lambda$. The terms $u_0^{b-1}X_0^{(b)}(r_0,u_0,M_0,\Lambda)$ represent the two-, three-, and four-loop contributions with longitudinal and transverse propagators $G_L(p)=(\overline{r_{0L}}+p^2)^{-1}$ and $G_T(p)=(\overline{r_{0T}}+p^2)^{-1}$, where

$$\overline{r_{0L}}=r_0+12u_0M_0^2,\quad\overline{r_{0T}}=r_0+4u_0M_0^2. \tag{18}$$

The two- and three-loop diagrams of $X_0^{(2)}$ and $X_0^{(3)}$ have been presented in Fig. 1 of Ref. [36]. The topology of the various vacuum diagrams of the four-loop contribution $X_0^{(4)}$ is shown in Fig. 1. There are (i) 12 types of diagrams (a–l) that are multiplicatively constructed from one-, two-, and three-loop diagrams and (ii) 14 topologically true four-loop vacuum diagrams (A–N). The specification of the propagators (either longitudinal or transverse) in A-N leads to 92 diagrams of the type (ii).

A. Mass shift and overall subtraction

We are primarily interested in the universal amplitude ratios A^+/A^- and R_ξ^+. They are independent of Λ, therefore we aim at a calculation of $\Gamma_0(r_0,u_0,M_0,\infty)$ in the limit $\Lambda\to\infty$ using the prescriptions of dimensional regularization. Accordingly we use the critical parameter r_{0c} in the dimensionally regularized form [57]

$$r_{0c}(u_0,\varepsilon)=u_0^{2/\varepsilon}S(\varepsilon), \tag{19}$$

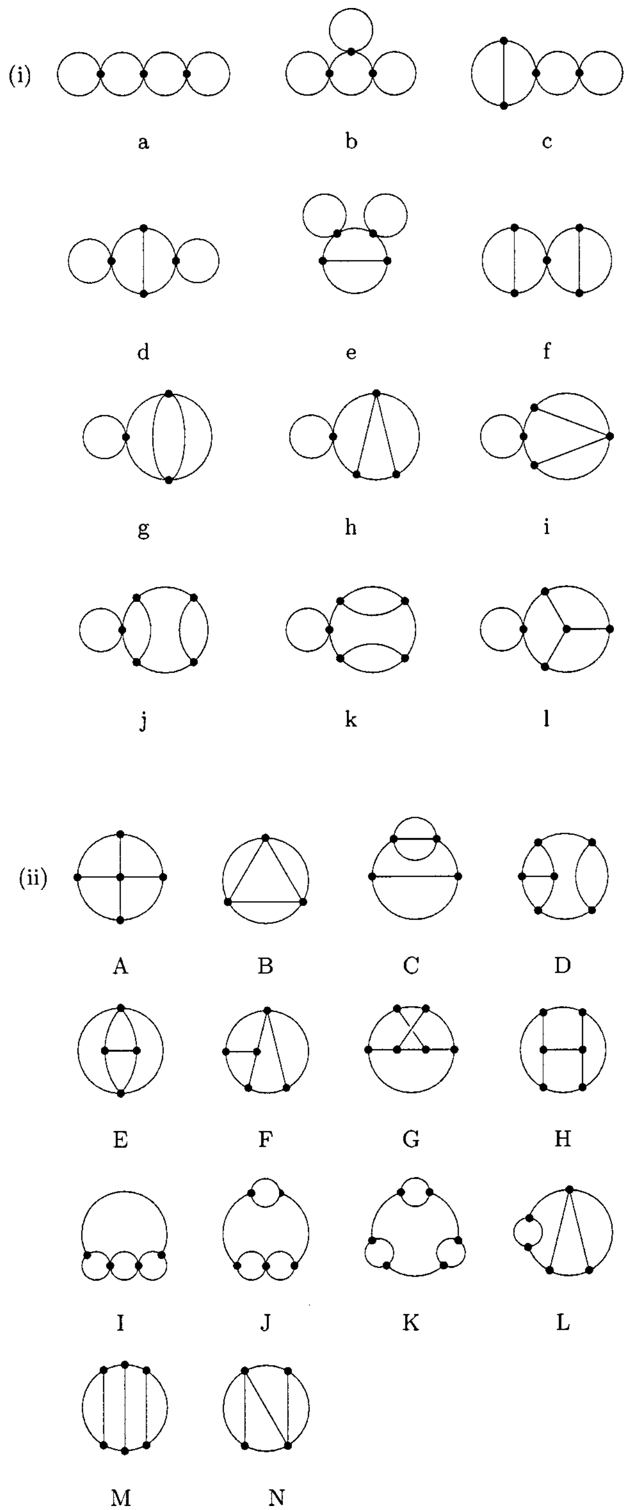

FIG. 1. Topology of vacuum diagrams in four-loop order determining the contribution $X_0^{(4)}$ to the Helmholtz free energy Γ_0, Eq. (17). (i) Four-loop graphs constructed from multiplication of lower-order diagrams, (ii) topologically true four-loop graphs. After specification of the propagators (longitudinal or transverse) one obtains the following numbers of different diagrams: 6 (A), 5 (B), 6 (C), 8 (D), 8 (E), 10 (F), 3 (G), 6 (H), 7 (I), 7 (J), 5 (K), 9 (L), 6 (M), 6 (N), thus a total number of 92 four-loop diagrams of the type (ii).

where the function $S(\varepsilon)$ is finite for $\varepsilon=4-d>0$ except for (simple) poles at $d_l=4-2/l$, $l=2,3,\ldots$. Near $d=d_l$, the structure of r_{0c} is

$$r_{0c}(u_0,\varepsilon)=u_0^{2/\varepsilon}\left[\frac{c_l(n)}{\varepsilon-2/l}+\widetilde{S}_l(\varepsilon,n)\right], \tag{20}$$

where $\widetilde{S}_l(\varepsilon,n)$ has a finite limit for $\varepsilon\to 2/l$. The pole at $d=3$ has the coefficient [34]

$$c_2(n)=\frac{n+2}{\pi^2}. \tag{21}$$

We do not use, however, the mass shift r_0-r_{0c} with r_{0c} in the form of Eqs. (19)–(21). As explained in earlier work [23,34], we use, near $d=3$, the simpler mass shift

$$r_0'(u_0,\varepsilon)=r_0-\delta r_0(u_0,\varepsilon), \tag{22}$$

$$\delta r_0(u_0,\varepsilon)=u_0^{2/\varepsilon}\left[\frac{c_2(n)}{\varepsilon-1}+C(n)\right]. \tag{23}$$

Here the function $\widetilde{S}_2(\varepsilon,n)$ has been replaced by the d-independent constant

$$C(n)=\frac{n+2}{\pi^2}\left(1-\gamma+\ln\frac{4\pi}{9}-2\ln 24\right), \tag{24}$$

where γ is Euler's constant. The parameter δr_0 contains the $d=3$ pole of r_{0c} but not the poles of r_{0c} at $d_l\neq 3$. For the choice of $C(n)$ we refer to Ref. [34]. [The description in terms of r_0' is only an intermediate step. The final results for the amplitude functions and amplitude ratios do not depend on the particular choice of δr_0 and of $C(n)$ [30], since the perturbation series will be expressed in terms of the correlation length.] Correspondingly, instead of $\overline{r_{0L}}$ and $\overline{r_{0T}}$, we use the longitudinal and transverse parameters

$$r_{0L}=r_0'+12u_0M_0^2, \quad r_{0T}=r_0'+4u_0M_0^2. \tag{25}$$

We are primarily interested in the singular part of the temperature dependence of the free energy at $d=3$. Therefore it is justified to subtract from Γ_0, regular contributions that consist of a term independent of r_0 and a term linear in r_0-r_{0c}. In the limit $\Lambda\to\infty$ these terms are divergent for $d>2$. In addition, Γ_0 contains a cutoff-dependent contribution proportional to $(r_0-r_{0c})^2$. We denote this fluctuation-induced regular contribution by

$$-\tfrac{1}{2}B_{cr}(\Lambda)t^2. \tag{26}$$

The coefficient $B_{cr}(\Lambda)$ has a finite (nonuniversal) value

$$B_{cr}\equiv B_{cr}(\infty) \tag{27}$$

in the limit $\Lambda\to\infty$ for $d<4$. (This value will be given in Sec. VI C.) All regular subtractions from Γ_0 that have a power-law dependence on Λ are ignored within dimensional regularization. As far as the term linear in r_0-r_{0c} is concerned, this is equivalent to setting $f_1=f_1^B$ in Eq. (3). The r_0-independent part of Γ_0 corresponding to f_0 in Eq. (3) contains a contribution in four-loop order [22,23] that has a logarithmically divergent cutoff dependence at $d=3$, which appears as a pole in the dimensional-regularization scheme. This four-loop pole contribution $P_4(u_0,d)$ to Γ_0 (which comes from diagrams of type B in Fig. 1) reads

$$P_4(u_0,d)=u_0^3\frac{n(n+2)(n+8)}{192\pi^2(d-3)}. \tag{28}$$

There exist no contributions to f_0 with $d=3$ pole terms beyond four-loop order, as can be seen from dimensional arguments. After substituting $r_0=r_0'+\delta r_0(u_0,\varepsilon)$ into Γ_0, we define the dimensionally regularized Helmholtz free energy as a function of r_0', u_0, and M_0 at $d=3$ by

$$\mathring{\Gamma}(r_0',u_0,M_0)=\lim_{d\to 3}[\Gamma_0(r_0'+\delta r_0,u_0,M_0,\infty)-\delta\Gamma_0(u_0,d)], \tag{29}$$

where the overall subtraction

$$\delta\Gamma_0(u_0,d)=P_4(u_0,d)+\frac{n(n+2)(n+8)}{192\pi^2}\times u_0^3\left[1-\gamma+\ln\pi-2\ln 24-\frac{21}{\pi^2}\zeta(3)\right] \tag{30}$$

contains the $d=3$ pole term (28) and an additional d-independent constant. Here we have chosen this constant so as to conform with the corresponding choice in the four-loop free energy of Refs. [23,33] for the special case $n=1$. This is equivalent to setting $f_{ns}(0)=f_0=f_0^B$ in Eq. (3). In summary we have the following identifications:

$$\lim_{h_0\to 0}\mathring{\Gamma}(r_0',u_0,M_0)=f_s^{\pm}(t)-\tfrac{1}{2}B_{cr}t^2 \tag{31}$$

and

$$\lim_{h_0\to 0}-\frac{\partial^2}{\partial t^2}\mathring{\Gamma}(r_0',u_0,M_0)=C_s^{\pm}(t)+B_{cr}, \tag{32}$$

$$C_{ns}(0)=B=C_B+B_{cr}, \tag{33}$$

apart from cutoff-dependent contributions.

B. Result in four-loop order

Our four-loop result can be written as

$$\mathring{\Gamma}(r_0',u_0,M_0)=\mathring{\Gamma}_{\mathrm{MF}}(r_0',u_0,M_0)+\sum_{b=1}^{4}\mathring{\Gamma}_b(r_0',u_0,M_0) \tag{34}$$

with the mean-field term

$$\mathring{\Gamma}_{\mathrm{MF}}(r_0',u_0,M_0)=\tfrac{1}{2}r_0'M_0^2+u_0M_0^4. \tag{35}$$

The contributions in b-loop order with $b=1,2,3,4$ have the form

$$\mathring{\Gamma}_b=\sum_{l=0}^{b-1}\sum_{k=0}^{1}(-1)^k2^{-l-k}F_{blk}(\bar{w},n)(24u_0)^{3-l}(M_0^2)^l \times\left[\frac{r_{0L}}{(24u_0)^2}\right]^{(4-b-2l)/2}\left(\ln\frac{r_{0L}}{(24u_0)^2}\right)^k. \tag{36}$$

The crucial information is contained in the analytic expression of the coefficients $F_{blk}(\bar{w},n)$ in b-loop order, which depend on the parameter

$$\bar{w}(r_0',u_0,M_0)=\frac{r_{0T}}{r_{0L}}=\frac{r_0'+4u_0M_0^2}{r_0'+12u_0M_0^2}. \tag{37}$$

For our purpose of deriving the bare Gibbs free energy (Sec. IV) at $h_0=0$ above T_c and near the coexistence curve below T_c, it is not necessary to calculate the complete $\bar{w}$ dependence of all coefficients $F_{blk}(\bar{w},n)$. Above T_c at $h_0=0$ we have $M_0^2=0$ and $\bar{w}=1$. Because of $M_0^2=0$ there are no contributions to $\mathring{\Gamma}$ with $l>0$. Thus, above T_c, we need only the coefficients F_{b00} and F_{b01} with $l=0$ at $\bar{w}=1$. Up to three-loop order $(b=1,2,3)$ these coefficients are given in Eqs. (11)–(16) of Ref. [36]. In four-loop order, the new coefficients $F_{blk}(\bar{w},n)$ with $l=0$ read, at $\bar{w}=1$ for general n in analytic form

$$F_{400}(1,n)=\frac{(4\pi)^{-4}}{1728}n(n+2)^2\left(\frac{1}{3}(n+2)+4\ln\frac{4}{3}\right), \tag{38}$$

$$F_{401}(1,n)=\frac{(4\pi)^{-4}}{864}n(n+2)\left(\frac{\pi^2}{6}(n+8)-2n-4\right). \tag{39}$$

For the special case $n=1$, these analytic expressions agree with the numerical values given in Table 2 of Ref. [23], Table 1 of Ref. [33], and Table A.1 of Ref. [27].

Below T_c, the parameter $\bar{w}$ becomes a small quantity near the coexistence curve where r_{0T} becomes small. If we employ an expansion of $\bar{w}$ with respect to u_0 at fixed $r_0'<0$ and at fixed small h_0 [compare Eqs. (82)–(86) of Sec. III], we obtain to leading order

$$\bar{w}(r_0',u_0,M_0)=\frac{1}{(-2r_0')}\left[\frac{h_0}{M_0}+3\pi^{-1}u_0(-2r_0')^{1/2}\right]+O(u_0^2,u_0(h_0/M_0)^{1/2}). \tag{40}$$

Thus, near the coexistence curve at finite M_0 below T_c, $\bar{w}$ can be considered as a small parameter and it suffices to calculate only the leading $\bar{w}$ dependence of $F_{blk}(\bar{w},n)$. The coefficients up to three-loop order $(b=1,2,3)$ are given in Eqs. (11)–(20) of Ref. [36]. The new four-loop coefficients near the coexistence curve below T_c read in analytic form for general n and for small $\bar{w}$

$$F_{400}(\bar{w},n)=\frac{(4\pi)^{-4}}{3456}\Bigg\{18+72\ln\frac{4}{3}+(n-1)\Bigg[-\frac{\bar{w}^{-3/2}}{3}+\bigg(4(n+1)\ln(4\bar{w})+8(n+2)\ln\frac{2}{3}+n^2+2n+7\bigg)\frac{1}{\bar{w}^{1/2}}+4(n+1)$$
$$\times\bigg(n+1-\frac{\pi^2}{12}(n+7)\bigg)\ln\bar{w}+\frac{2\pi^2}{3}\bigg(n(n+8)\ln\frac{2}{3}+15\ln 2-7\ln 3\bigg)+\frac{2n^3}{3}+2n^2+10n+\frac{38}{3}$$
$$+7(n^2+11n+13)\zeta(3)-8(n^2+3n+5)\ln 3+8(2n^2+5n+6)\ln 2+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg]\Bigg\}, \quad (41)$$

$$F_{401}(\bar{w},n)=\frac{(4\pi)^{-4}}{1728}\Bigg\{9\pi^2-36-(n-1)\Bigg[\frac{4(n+2)}{\bar{w}^{1/2}}+4(n^2+3n+5)-(n^2+11n+27)\frac{\pi^2}{3}+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg]\Bigg\}, \quad (42)$$

$$F_{410}(\bar{w},n)=-\frac{(4\pi)^{-4}}{2592}\Bigg\{\Lambda_1(n)+(n-1)\Bigg[\frac{1}{\bar{w}^{3/2}}-\frac{n^2-n+11-8(n-1)\ln 3}{\bar{w}^{1/2}}+(n+1)\bigg(4(n+2)-\frac{\pi^2}{2}(n+7)\bigg)\ln\bar{w}$$
$$+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg]\Bigg\}, \quad (43)$$

$$\Lambda_1(n)=36+360\ln 5-72\ln 3-576\ln 2-81\bigg(\ln\frac{3}{4}\bigg)^2-162\bigg[\mathrm{Li}_2\bigg(-\frac{1}{4}\bigg)+\mathrm{Li}_2\bigg(-\frac{2}{3}\bigg)+\mathrm{Li}_2\bigg(-\frac{1}{3}\bigg)\bigg]-\frac{81}{2}\pi^2+324c_4+648J_{1,1}^{(1)}$$
$$+648E_1+(n-1)\bigg[(3-\pi^2\ln 3+16\ln 2)n^2+\bigg(\frac{10}{3}\pi^2+\frac{51}{2}\zeta(3)-27+56\ln 2-36\ln 3-\frac{11}{6}\pi^2\ln 3-7\pi^2\ln 2$$
$$+\frac{1}{6}(\ln 3)^3-2\,\mathrm{polylog}\bigg(3,\frac{1}{3}\bigg)+\mathrm{polylog}\bigg(3,-\frac{1}{3}\bigg)\bigg)n+\frac{82}{3}+\frac{\pi^2}{2}-102\ln 3+240\ln 2+\frac{167}{2}\zeta(3)+38\mathrm{Li}_2\bigg(-\frac{1}{3}\bigg)$$
$$+34\mathrm{Li}_2\bigg(\frac{1}{3}\bigg)-45(\ln 2)^2+33(\ln 3)^2+22(\ln 2)(\ln 3)+\frac{40}{3}(\ln 3)^3+\frac{37}{3}\pi^2\ln 3-32\pi^2\ln 2-106\,\mathrm{polylog}\bigg(3,\frac{1}{3}\bigg)$$
$$+26\,\mathrm{polylog}\bigg(3,-\frac{1}{3}\bigg)+144J_{1,1}^{(0)}+X_{410}\bigg] \quad (44)$$

$$=\tfrac{1}{4}(12.989\,945\,n^3-143.974\,71\,n^2+986.035\,48\,n-556.259\,06), \quad (45)$$

$$F_{411}(\bar{w},n)=-\frac{(4\pi)^{-4}}{648}\Bigg\{9+(n-1)\Bigg[2(n+2)\frac{1}{\bar{w}^{1/2}}-3(n+1)+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg]\Bigg\}, \quad (46)$$

$$F_{420}(\bar{w},n)=\frac{(4\pi)^{-4}}{1944}\Bigg\{\Lambda_2(n)-(n-1)\Bigg(\frac{1}{\bar{w}^{3/2}}+\frac{3}{\bar{w}^{1/2}}\bigg[(n+1)\ln\bar{w}+\frac{2n}{3}-3+4n\ln 2+6\ln\frac{2}{3}\bigg]$$
$$-(n+1)\bigg[n+5-\frac{\pi^2}{4}(n+7)\bigg]\ln\bar{w}+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg)\Bigg\}, \quad (47)$$

$$\Lambda_2(n)=\frac{27}{4}-\frac{567}{16}\pi^2+243\ln\frac{4}{3}-\frac{2673}{4}\mathrm{Li}_2\bigg(-\frac{1}{3}\bigg)-243c_1+486c_4+1944F_1+972A_1+1944J_{2,1}^{(1)}-(n-1)\Bigg\{\bigg[\frac{\pi^2}{4}-1-4\ln 2$$
$$+\frac{\pi^2}{2}\ln 3\bigg]n^2+\bigg[\frac{173}{144}\pi^2-\frac{337}{12}-\frac{101}{4}\zeta(3)-26\ln 2+6\ln 3-\frac{43}{12}\pi^2\ln 3+16c_3+19\bigg(\frac{\pi^2}{2}\ln 2-\frac{(\ln 3)^3}{12}+\mathrm{polylog}\bigg(3,\frac{1}{3}\bigg)$$
$$-\frac{1}{2}\mathrm{polylog}\bigg(3,-\frac{1}{3}\bigg)\bigg)\bigg]n+\frac{1379}{60}+\frac{883}{72}\pi^2-\frac{433}{4}\zeta(3)-100\ln 3+\frac{2773}{15}\ln 2-20\mathrm{Li}_2\bigg(-\frac{1}{3}\bigg)-\frac{647}{4}\mathrm{Li}_2\bigg(\frac{1}{3}\bigg)$$
$$-\frac{871}{8}(\ln 3)^2+192(\ln 2)(\ln 3)+44\pi^2\ln 2-\frac{121}{6}\pi^2\ln 3-\frac{62}{3}(\ln 3)^3-43\,\mathrm{polylog}\bigg(3,-\frac{1}{3}\bigg)+167\,\mathrm{polylog}\bigg(3,\frac{1}{3}\bigg)$$

$$+7c_2+16c_3-216(J_{2,1}^{(0)}+J_{1,2}^{(0)})-X_{420}\Bigg\} \tag{48}$$

$$=\tfrac{1}{4}(-16.464\,99\,n^3+80.616\,37\,n^2-125.3736\,n+256.6283), \tag{49}$$

$$F_{430}(\bar{w},n)=-\frac{(4\pi)^{-4}}{1458}\Bigg\{\Lambda_3(n)+(n-1)\Bigg[\frac{1}{3\bar{w}^{3/2}}-\Bigg((n+1)\ln\bar{w}+n+5+4n\ln 2+6\ln\frac{2}{3}\Bigg)\frac{1}{\bar{w}^{1/2}}+(n+1)\Bigg(1-\frac{\pi^2}{24}(n+7)\Bigg)$$

$$\times\ln\bar{w}+O(\bar{w}^{1/2},\bar{w}^{1/2}\ln\bar{w})\Bigg]\Bigg\}, \tag{50}$$

$$\Lambda_3(n)=81\ln\frac{4}{3}-\frac{27}{2}\pi^2-\frac{1053}{4}\mathrm{Li}_2\Bigg(-\frac{1}{3}\Bigg)+243c_4+729J_{2,2}^{(1)}+1458D_1+162G_1+972H_1+(n-1)\Bigg[-\Bigg(\frac{\pi^2}{16}+\frac{\pi^2}{12}\ln 3\Bigg)n^2$$

$$+\Bigg(\frac{23}{3}-\frac{59}{72}\pi^2+4\ln 2+\frac{399}{64}\zeta(3)-5c_3+\frac{35}{64}(\ln 3)^3-\frac{13}{4}\pi^2\ln 2+\frac{419}{192}\pi^2\ln 3-\frac{105}{16}\mathrm{polylog}\Bigg(3,\frac{1}{3}\Bigg)+\frac{105}{32}\mathrm{polylog}$$

$$\times\Bigg(3,-\frac{1}{3}\Bigg)\Bigg)n+\frac{643}{45}+\frac{817}{144}\pi^2+\frac{3489}{64}\zeta(3)+\frac{493}{15}\ln 2-\frac{65}{2}\ln 3-\frac{26}{3}\mathrm{Li}_2\Bigg(-\frac{1}{3}\Bigg)-\frac{1183}{12}\mathrm{Li}_2\Bigg(\frac{1}{3}\Bigg)-\frac{465}{8}(\ln 3)^2$$

$$+\frac{152}{3}(\ln 2)(\ln 3)+\frac{613}{64}(\ln 3)^3+\frac{1285}{192}\pi^2\ln 3-\frac{55}{3}\pi^2\ln 2-2c_2+5c_3+\frac{543}{32}\mathrm{polylog}\Bigg(3,-\frac{1}{3}\Bigg)$$

$$-\frac{1191}{16}\mathrm{polylog}\Bigg(3,\frac{1}{3}\Bigg)+162J_{2,2}^{(0)}+X_{430}\Bigg] \tag{51}$$

$$=\tfrac{1}{4}(-6.082\,n^3+30.39\,n^2+20.87\,n-1.4655). \tag{52}$$

The numerical values of $\Lambda_1(n)$–$\Lambda_3(n)$ in Eqs. (44), (48) and (51) have been obtained using Eqs. (A1)–(A48) of Appendix A.

All other coefficients F_{4lk} vanish. The analytic expressions for the constants c_1–c_4, A_1–H_1, X_{410}–X_{430}, and $J_{m,n}^{(k)}$ and the definitions of the functions $\zeta(n)$, $\mathrm{Li}_2(x)$, and $\mathrm{polylog}(n,x)$ are given in Appendix A.

A detailed derivation of the four-loop coefficients F_{4lk} is given in Ref. [41]. The coefficients with $k=1$ in Eq. (36) correspond to logarithms with respect to the coupling u_0, which, for $\varepsilon\to 1$, arise from the nonanalytic u_0 dependence of δr_0, Eq. (23), as expected. Equations (34)–(52) are the starting point of the present paper. They provide the basis for deriving the analytic form of the amplitude functions of the Gibbs free energy and the specific heat for general n above and below T_c.

For the special case $n=1$ there is no dependence of F_{blk} on $\bar{w}$. For this case, Eqs. (41)–(52) agree with the numerical values given in Table 2 of Ref. [23], Table 1 of Ref. [33], and Table A.1 of Ref. [27]. The terms proportional to $n-1$ due to the transverse order-parameter fluctuations depend nonanalytically on r_{0T} through $\bar{w}^{-3/2}$, $\bar{w}^{-1}$, $\bar{w}^{-1/2}\ln\bar{w}$, $\bar{w}^{-1/2}$, and $\ln\bar{w}$. [Higher orders in $\bar{w}$ such as $O(\bar{w}^{1/2})$ and $O(\bar{w}^{1/2}\ln\bar{w})$ that yield a vanishing contribution in the limit $\bar{w}\to 0$ have been neglected in Eqs. (41)–(50) because they do not contribute to the specific heat for $h_0\to 0$ below T_c.] It is obvious that at this stage of the perturbation theory it is not possible to set $h_0=0$ in $\bar{w}$, Eq. (40), since this would yield nonanalytic dependences $\sim u_0^{-3/2}, u_0^{-1/2}$ of the coefficients F_{blk}. The nonanalytic $\bar{w}$ dependence will lead to perturbative contributions of $\mathring{\Gamma}_4$ to the Gibbs free energy, which diverge when the coexistence curve is approached ($T<T_c$, $h_0\to 0$). This is again the effect of the Goldstone modes that was found previously in two- [34] and three-loop order [36]. For $O(n)$ symmetric quantities, however, such divergences should cancel among themselves [58]. This is indeed the case, at least up to four-loop order, for the *complete* perturbative results of the Gibbs free energy and the specific heat as we shall see in Sec. IV.

III. CORRELATION LENGTHS

The bare perturbative expression (36) contains terms (with $k=1$) that have a logarithmic dependence on u_0, as expected [22,23,30], because of the nonanalytic u_0 dependence of δr_0, Eq. (23). In order to obtain Borel-resummable series, we shall rewrite the free energy as a function of appropriately defined correlation lengths ξ_+ and ξ_- [30,31] that absorb the logarithmic terms, except for one special four-loop term to be discussed later. For this purpose we calculate the relation between r_0' and $\xi_\pm$ up to four-loop order in analytic form.

A. Above T_c

Above T_c, the square of the (second moment) correlation length is defined via

$$\xi_+^2 = \mathring{\chi}_+(0)\,\partial\mathring{\chi}_+(q)^{-1}/\partial q^2|_{q=0}, \tag{53}$$

where $\mathring{\chi}_+(q)^{-1} = \Gamma_0^{(0,2)}(q,r_0,u_0)$ is the inverse susceptibility at finite wave number q. The two-point vertex function $\Gamma_0^{(0,2)}(q,r_0,u_0)$ is given by

$$\Gamma_0^{(0,2)}(q,r_0,u_0) = r_0 + q^2 - \Sigma_0(q,r_0,u_0), \tag{54}$$

where $\Sigma_0(q,r_0,u_0)$ is the self-energy. The diagrammatic contributions to Σ_0 up to two- and three-loop order have been given in Refs. [34,36], respectively. The diagrammatic four-loop contributions are given in Appendix B. This determines $\xi_+ = \xi_+(r_0,u_0,d)$ up to $O(u_0^4)$. The inverse function has the power series [30]

$$r_0(\xi_+,u_0,d) = \xi_+^{-2} R_{0+}(u_0\xi_+^{\varepsilon},\varepsilon) = \xi_+^{-2}\left[1 + \sum_{m=1}^{\infty} \frac{b_m(d)}{1-m\varepsilon/2}(u_0\xi_+^{\varepsilon}/\varepsilon)^m\right], \tag{55}$$

where $b_m(d)$ are the expansion coefficients of the function

$$\mathring{P}_+(u_0\xi_+^{\varepsilon},d) = \left(\frac{\partial r_0}{\partial \xi_+^{-2}}\right)_{u_0} = \sum_{m=0}^{\infty} b_m(d)(u_0\xi_+^{\varepsilon})^m. \tag{56}$$

The coefficients $b_m(d)$ are finite for $d=3$. They are known up to $m=6$ in numerical form for $n=1,2,3$ [32]. Here we shall present their analytic form for general n up to $m=4$. In order to subtract the $d=3$ pole in Eq. (55), we consider, for $r_0>0$,

$$\lim_{d\to 3}[r_0(\xi_+,u_0,d) - \delta r_0(u_0,\varepsilon)] \equiv r_0'(\xi_+,u_0), \tag{57}$$

where $\delta r_0(u_0,\varepsilon)$ is defined in Eq. (23). From Appendix B we obtain the function $r_0'(\xi_+,u_0)>0$ up to four-loop order,

$$r_0'(\xi_+,u_0) = \xi_+^{-2}\Bigg\{1 + \frac{n+2}{\pi}u_0\xi_+ + \frac{n+2}{\pi^2}(u_0\xi_+)^2 \times\left[\frac{1}{27} + 2\ln(24u_0\xi_+)\right] - 2b_3(u_0\xi_+)^3 - b_4(u_0\xi_+)^4 + O(u_0^5\xi_+^5)\Bigg\}, \tag{58}$$

with the three-loop coefficient [36]

$$b_3 = \frac{n+2}{\pi^3}\left[\lambda_3 - \frac{n}{12} - \frac{11}{18}\right], \tag{59}$$

$$\lambda_3 = \frac{43n+182}{54}\ln\frac{3}{4} + \frac{4}{3}(n+8)\left[\mathrm{Li}_2\left(-\frac{1}{3}\right) + \frac{\pi^2}{12}\right] \tag{60}$$

and the new four-loop coefficient

$$b_4 = -\frac{n+2}{\pi^4}\left[\lambda_4 + \frac{2n^2}{27} + \frac{1268}{729}n + \frac{1672}{729}\right], \tag{61}$$

with

$$\begin{aligned}\lambda_4 = {}& \frac{16}{3}(5n+22)\left(J_{1,1}^{(1)} - \frac{1}{2}J_{2,1}^{(1)} + J_{3,1}^{(1)} + \frac{3}{4}E_1 - \frac{1}{4}E_1' + E_1''\right) - \frac{1}{27}(173n+178)\left[\mathrm{Li}_2\left(-\frac{1}{4}\right) + \mathrm{Li}_2\left(-\frac{2}{3}\right) + \frac{1}{2}\left(\ln\frac{3}{4}\right)^2\right]\\ & + \frac{1}{9}(5n+22)\left[\mathrm{Li}_2\left(\frac{1}{3}\right) - \mathrm{Li}_2\left(\frac{1}{6}\right) - \frac{1}{2}(\ln 2)^2\right] + 4(n^2+6n+20)c_4 + \left(\frac{32}{27}\ln\frac{4}{3} - \frac{41}{216}\pi^2 - \frac{32}{9}\mathrm{Li}_2\left(-\frac{1}{3}\right)\right)n^2\\ & + \left(-\frac{70}{27}\pi^2 + \frac{404}{27}\ln\frac{3}{4} + \frac{797}{36}\ln\frac{5}{3} - \frac{740}{27}\mathrm{Li}_2\left(-\frac{1}{3}\right)\right)n - \frac{275}{54}\pi^2 + \frac{104}{3}\ln\frac{3}{4} + \frac{257}{6}\ln\frac{5}{3} - \frac{1864}{27}\mathrm{Li}_2\left(-\frac{1}{3}\right).\end{aligned} \tag{62}$$

For $n=1,2,3$ the numerical values of b_4 agree with those given in Table 2 of Ref. [32]. The analytic expressions for the constants E_1' and E_1'' are given in Appendix A.

We shall also need the function [30]

$$\mathring{h}(\xi_+,u_0,d) = r_0(\xi_+,u_0,d) - r_{0c}(u_0,\varepsilon), \tag{63}$$

which differs from $r_0'(\xi_+,u_0,d)$ only by a ξ_+ independent constant,

$$\mathring{h}(\xi_+,u_0,d) = r_0'(\xi_+,u_0,d) + \delta r_0(u_0,\varepsilon) - r_{0c}(u_0,\varepsilon). \tag{64}$$

For $d\to 3$ the function $\mathring{h}$ reads

$$\mathring{h}(\xi_+,u_0,3) = r_0'(\xi_+,u_0) + u_0^2[C(n) - \tilde{S}_2(1,n)], \tag{65}$$

where $r_0'(\xi_+,u_0)$ is given by Eqs. (58)–(62) and $\tilde{S}_2(1,n)$ is defined in Eq. (20).

B. Below T_c

Below T_c we need to express $r_0'<0$ in terms of the pseudocorrelation length ξ_- as defined in Ref. [31]. For the procedure of obtaining the function $r_0'(\xi_-,u_0,d)<0$ below T_c from the known function $R_{0+}(u_0\xi_+^\varepsilon,\varepsilon)$ above T_c see also Appendix A of Ref. [33] and Appendix A of Ref. [34]. This yields the following analytic four-loop result for $r_0'<0$ in three dimensions,

$$-2r_0'(\xi_-,u_0)=\xi_-^{-2}\Bigg\{1+\frac{n+2}{\pi}u_0\xi_- -\frac{n+2}{\pi^2}(u_0\xi_-)^2\times\left[\frac{1385}{108}+4\ln(24u_0\xi_-)\right]-2a_{r3}(u_0\xi_-)^3-2a_{r4}(u_0\xi_-)^4+O(u_0^5\xi_-^5)\Bigg\} \tag{66}$$

with the three-loop coefficient [36]

$$a_{r3}=-\frac{n+2}{\pi^3}\left[2\lambda_3+\frac{73}{12}n+\frac{4349}{72}\right] \tag{67}$$

and the new four-loop coefficient

$$a_{r4}=\frac{n+2}{\pi^4}\left[\lambda_4+\frac{43\,145}{6912}n^2+\frac{16\,374\,535}{93\,312}n+\frac{38\,265\,055}{46\,656}+\frac{3}{16}(3n^2+50n+244)\zeta(3)+\frac{\pi^4}{80}(5n+22)\right]. \tag{68}$$

The numerical values of a_{r4} for $n=1$, 2, 3 are

$$a_{r4}=34.5540703 \quad (n=1), \tag{69}$$

$$a_{r4}=54.9501255 \quad (n=2), \tag{70}$$

$$a_{r4}=80.5039152 \quad (n=3). \tag{71}$$

For $n=1$, the numerical value of a_{r4} agrees with that given in Table 2 of Ref. [33].

Equations (58)–(62) and (66)–(68) will be needed to express the free energy in terms of ξ_+ and ξ_- in Sec. IV and to define Borel-resummable amplitude functions in Sec. V.

IV. BARE GIBBS FREE ENERGY

In order to calculate the specific heat we first need to derive the Gibbs free energy. According to Eq. (16), the bare Gibbs free energy $\mathring{\mathcal{F}}$ is determined by the bare Helmholtz free energy $\mathring{\Gamma}$ and the order parameter M_0 as

$$\mathring{\mathcal{F}}(r_0',u_0,h_0)=\mathring{\Gamma}(r_0',u_0,M_0(r_0',u_0,h_0))-h_0M_0(r_0',u_0,h_0). \tag{72}$$

Above T_c at $h_0=0$ it is straightforward to derive the perturbative expression of $\mathring{\mathcal{F}}_+$ from Eq. (34) for $M_0=0$ up to four-loop order as

$$\mathring{\mathcal{F}}_+(r_0',u_0)\equiv\mathring{\Gamma}(r_0',u_0,0)=-\frac{n}{12\pi}r_0'^{3/2}+\frac{n(n+2)}{(4\pi)^2}u_0r_0'-\frac{2n(n+2)}{(4\pi)^3}\times u_0^2r_0'^{1/2}\left[n-6-8\ln\frac{3}{4}+4\ln\frac{r_0'}{(24u_0)^2}\right]+\frac{8n(n+2)}{(4\pi)^4}u_0^3\left[\frac{(n+2)^2}{3}+4(n+2)\times\ln\frac{4}{3}+\left(2n+4-\frac{\pi^2}{6}(n+8)\right)\ln\frac{r_0'}{(24u_0)^2}\right]+O(u_0^4,u_0^4\ln u_0) \tag{73}$$

for $r_0'>0$. To obtain the $O(u_0^3)$ term we have used the new four-loop coefficients, Eqs. (38) and (39). The terms up to $O(u_0^2)$ are identical with the previous (corrected) three-loop result for $\mathring{\mathcal{F}}_+$ [36].

Below T_c one expects on general grounds [58] that the Gibbs free energy $\mathring{\mathcal{F}}$ should be free of Goldstone singularities and that a finite limit

$$\mathring{\mathcal{F}}_-(r_0',u_0)\equiv\lim_{h_0\to 0}\mathring{\mathcal{F}}(r_0',u_0,h_0) \tag{74}$$

should exist for $r_0'<0$ and for general n. Because of spurious Goldstone singularities appearing in the four-loop term $\mathring{\Gamma}_4$ of the Helmholtz free energy $\mathring{\Gamma}(r_0',u_0,M_0)$, some care is necessary in deriving the correct perturbative expression of $\mathring{\mathcal{F}}_-(r_0',u_0)$ up to four-loop order corresponding to $O(u_0^3,u_0^3\ln u_0)$. The second term on the right-hand side (rhs) of Eq. (72) does not contribute in the limit $h_0\to 0$ at fixed $r_0'<0$. In the first term $\mathring{\Gamma}(r_0',u_0,M_0)$, it is necessary to substitute $M_0(r_0',u_0,h_0)$ in an appropriate perturbative form.

In the following, we distinguish two types of $O(u_0^3,u_0^3\ln u_0)$ contributions to the Gibbs free energy: (i) those that are obtained from the known Helmholtz free energy up to three-loop order [36] by substituting M_0 up to sufficiently high order, (ii) those that come directly from the new four-loop term $\mathring{\Gamma}_4(r_0',u_0,M_0)$ of the Helmholtz free energy with M_0 replaced by its lowest-order form.

A. Contributions of $O(u_0^3)$ from $\mathring{\Gamma}$ up to three-loop order

As far as step (i) is concerned we first consider the mean-field part $\mathring{\Gamma}_{\mathrm{MF}}$ and the one-loop part $\mathring{\Gamma}_1$ of $\mathring{\Gamma}$, Eqs. (34)–(36),

$$\mathring{\Gamma}_{\rm MF}(r_0',u_0,M_0)+\mathring{\Gamma}_1(r_0',u_0,M_0) = \frac{1}{2}r_0'M_0^2+u_0M_0^4-\frac{(r_0'+12u_0M_0^2)^{3/2}}{12\pi} -(n-1)\frac{(r_0'+4u_0M_0^2)^{3/2}}{12\pi}, \qquad (75)$$

where we have used $F_{100}(\overline{w},n)$ given in Eq. (11) of Ref. [36]. We shall show that in the perturbative form of M_0^2 to be substituted into Eq. (75), it suffices to keep the terms only up to *two-loop* order in order to obtain the contributions of Eq. (75) to $\mathring{\mathcal{F}}_-$ up to four-loop order.

Unlike the last term on the right-hand side of Eq. (75), the square of the order parameter in the first three terms can be substituted directly at $h_0=0$, where $M_0(r_0',u_0,0)^2$ has the expansion

$$M_0(r_0',u_0,0)^2=\frac{(-2r_0')}{8u_0}+\frac{3}{4\pi}(-2r_0')^{1/2}+\Delta M_0^2. \qquad (76)$$

Here the leading contribution to ΔM_0^2 is the two-loop term [34]

$$\Delta M_0^2=-\frac{u_0}{2\pi^2}(n+2)\ln\frac{(-2r_0')^{1/2}}{24u_0}+\frac{u_0}{8\pi^2} \times[10-n+4(n-1)\ln 3]+O(u_0^2,u_0^2\ln u_0). \qquad (77)$$

Substituting Eq. (76) into the first three terms on the right-hand side of Eq.(75) we get

$$\frac{1}{2}r_0'M_0^2=-\frac{(r_0')^2}{8u_0}+\frac{1}{2}r_0'\left[\frac{3}{4\pi}(-2r_0')^{1/2}+\Delta M_0^2\right], \qquad (78)$$

$$u_0M_0^4=\frac{(r_0')^2}{16u_0}-\frac{1}{2}r_0'\left[\frac{3}{4\pi}(-2r_0')^{1/2}+\Delta M_0^2\right]+\frac{9u_0}{16\pi^2}(-2r_0') +\frac{3u_0}{2\pi}(-2r_0')^{1/2}\Delta M_0^2+u_0(\Delta M_0^2)^2, \qquad (79)$$

$$-\frac{(r_0'+12u_0M_0^2)^{3/2}}{12\pi}=-\frac{1}{12\pi}\left[-2r_0'+\frac{9u_0}{\pi}(-2r_0')^{1/2}\right]^{3/2} -\frac{3}{2\pi}u_0\left[-2r_0'+\frac{9u_0}{\pi}(-2r_0')^{1/2}\right]^{1/2} \times\Delta M_0^2+O(u_0^2[\Delta M_0^2]^2). \qquad (80)$$

In the sum of Eqs. (78)–(80), the terms of $O(\Delta M_0^2)$ and $O(u_0\Delta M_0^2)$ cancel out, but the terms of $O(u_0[\Delta M_0^2]^2)$ and $O(u_0^2\Delta M_0^2)$ contribute to the four-loop term of $\mathcal{F}_-(r_0,u_0)$, which is of $O(u_0^3,u_0^3\ln u_0)$. This implies that in the sum

$$\frac{1}{2}r_0'M_0^2+u_0M_0^4-\frac{(r_0'+12u_0M_0^2)^{3/2}}{12\pi} =-\frac{(-2r_0')^2}{64u_0}-\frac{1}{12\pi}(-2r_0')^{3/2}-\frac{9}{16\pi^2}u_0(-2r_0') -\frac{81}{32\pi^3}u_0^2(-2r_0')^{1/2}+\frac{243}{64\pi^4}u_0^3+u_0(\Delta M_0^2)^2 -\frac{27}{4\pi^2}u_0^2\Delta M_0^2+O(u_0^4,u_0^3\Delta M_0^2,u_0^2[\Delta M_0^2]^2), \qquad (81)$$

the two-loop part of the term ΔM_0^2, Eq. (77), contributes to $\mathring{\mathcal{F}}_-$ in four-loop order, but not the three-loop term of ΔM_0^2.

The last term on the right-hand side of Eq. (75) must not be treated directly at $h_0=0$ but requires an expansion of $(r_0'+4u_0M_0^2)^{3/2}$ in powers of u_0 at finite h_0. The starting point is the expansion

$$M_0(r_0',u_0,h_0)^2=\frac{-r_0'+\mathring{\chi}_T^{-1}}{4u_0}+\frac{3}{4\pi}(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2} +\frac{n-1}{4\pi}\mathring{\chi}_T^{-1/2}+\Delta M_0^2(h_0), \qquad (82)$$

$$\Delta M_0^2(h_0)=\frac{u_0}{8\pi^2}\left\{(n-1)\left[6w^{1/2}+\frac{3w}{1+2w^{1/2}} -4\ln\frac{1+2w^{1/2}}{3}\right]+10-n+9w\right\} -\frac{u_0}{2\pi^2}(n+2)\ln\frac{(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}}{24u_0} +O(u_0^2,u_0^2\ln u_0), \qquad (83)$$

$$w=\mathring{\chi}_T^{-1}(-2r_0'+3\mathring{\chi}_T^{-1})^{-1}, \qquad (84)$$

$$\mathring{\chi}_T=M_0/h_0, \qquad (85)$$

as given in Eqs. (34)–(36) of Ref. [34]. Note that this expanded form of M_0^2 is still an implicit equation for M_0 as a function of r_0' and h_0. Here the leading contribution to $\Delta M_0^2(h_0)$, Eq. (83), is of two-loop order. Equation (82) implies

$$r_0'+4u_0M_0^2=\mathring{\chi}_T^{-1}+4u_0\left\{\frac{3}{4\pi}(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}+\frac{1}{4\pi} \times(n-1)\mathring{\chi}_T^{-1/2}+\Delta M_0^2(h_0)\right\}. \qquad (86)$$

Substituting Eq. (86) into the last term of Eq. (75) and expanding up to $O(u_0^3)$ yields

$$-\frac{n-1}{12\pi}(r_0'+4u_0M_0^2)^{3/2}=-\frac{n-1}{12\pi}\Bigg[\mathring{\chi}_T^{-3/2}+\frac{3u_0}{2\pi}\mathring{\chi}_T^{-1/2}[3(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}+(n-1)\mathring{\chi}_T^{-1/2}]+\frac{3u_0^2}{8\pi^2}\mathring{\chi}_T^{1/2}[3(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}+(n-1)\mathring{\chi}_T^{-1/2}]^2-\frac{u_0^3}{16\pi^3}\mathring{\chi}_T^{3/2}[3(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}+(n-1)\mathring{\chi}_T^{-1/2}]^3+6u_0\mathring{\chi}_T^{-1/2}\Delta M_0^2(h_0)+\frac{3u_0^2}{\pi}\mathring{\chi}_T^{1/2}[3(-2r_0'+3\mathring{\chi}_T^{-1})^{1/2}+(n-1)\mathring{\chi}_T^{-1/2}]\Delta M_0^2(h_0)\Bigg]+O(u_0^4,u_0^3\Delta M_0^2(h_0),u_0^2[\Delta M_0^2(h_0)]^2). \tag{87}$$

From the term $u_0\mathring{\chi}_T^{-1/2}\Delta M_0^2(h_0)$ we see that, for finite h_0 corresponding to finite $\mathring{\chi}_T^{-1/2}$, the three-loop term of $\Delta M_0^2(h_0)$ contributes to the four-loop free energy $\mathring{\mathcal{F}}_-'(r_0',u_0,h_0)$. In the limit $h_0\to 0$, however, the term $u_0\mathring{\chi}_T^{-1/2}\Delta M_0^2(h)$ vanishes. In the last term on the right-hand side of Eq. (87), which is proportional to $\mathring{\chi}_T^{1/2}u_0^2\Delta M_0^2(h_0)$, it suffices to substitute $\Delta M_0^2(h_0)$ in two-loop order. This term diverges for $\mathring{\chi}_T\to\infty$, in addition to other divergent terms $\sim\mathring{\chi}_T^{3/2}$ and $\sim\mathring{\chi}_T^{1/2}$ in Eq. (87). These divergences are canceled by the higher-order contributions $\mathring{\Gamma}_2$, $\mathring{\Gamma}_3$, and $\mathring{\Gamma}_4$ of the Helmholtz free energy, after substituting M_0 in two-, one-, or zero-loop order, respectively. Here we skip the details of these lengthy calculations. In summary, inserting Eq. (82) into $\mathring{\Gamma}$, Eq. (34), up to three-loop order corresponding to step (i), leads to the contribution at small finite h_0 below T_c:

$$\mathring{\Gamma}_{\rm MF}+\mathring{\Gamma}_1+\mathring{\Gamma}_2+\mathring{\Gamma}_3=\mathring{\mathcal{F}}_-(r_0',u_0)^{\text{three-loop}}+\frac{u_0^3}{(4\pi)^4}\Bigg\{(n-1)\Bigg[\frac{36}{w^{3/2}}-\frac{12}{w^{1/2}}\Bigg(n^2+41+4(n+2)\ln\frac{-2r_0'}{(24u_0)^2}-8(n-1)\ln 3\Bigg)\Bigg]-\frac{8n^4}{3}+\frac{8n^3}{3}-452n^2+\frac{25\,048}{15}n-\frac{6928}{15}+(15n^2+63n+111)\pi^2+1296c_1-48(n-1)c_2+2268\mathrm{Li}_2\left(-\frac{1}{3}\right)-324(n-1)\mathrm{Li}_2\left(\frac{1}{3}\right)-16(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\Bigg[(n+2)\ln\frac{-2r_0'}{(24u_0)^2}-4(n-1)\ln 3+n(n-1)\Bigg]-(64n^2+34n-98)(\ln 3)^2+16(2n^3-4n^2+2n+135)\ln 3+\frac{144}{5}(37n-187)\ln 2\Bigg\}+O(w^{1/2},w^{1/2}\ln w,u_0^4,u_0^4\ln u_0), \tag{88}$$

where

$$\mathring{\mathcal{F}}_-(r_0',u_0)^{\text{three-loop}}=-\frac{1}{64u_0}(-2r_0')^2-\frac{1}{12\pi}(-2r_0')^{3/2}-\frac{u_0}{(4\pi)^2}(-2r_0')\Bigg[6+2(n-1)\ln 3-(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\Bigg]+\frac{u_0^2}{(4\pi)^3}(-2r_0')^{1/2}\Bigg[\frac{8}{3}(11n+7)-216c_1-8(n-1)c_2+\frac{8}{3}(31n+95)\ln 2-8(4n+17)\ln 3-21(n-1)\times\Bigg[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\Bigg]+\frac{\pi^2}{2}(3n^2+11n-5)+54\mathrm{Li}_2\left(-\frac{1}{3}\right)+16(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\Bigg] \tag{89}$$

is identical with the previous three-loop result, Eq. (32) of Ref. [36], where it was written in a slightly different form, see our Eqs. (A54) and (A55). In the $O(u_0^3)$ contribution of Eq. (88), there are terms $\sim w^{-3/2}$ and $w^{-1/2}$ that would diverge in the limit $h_0\to 0$ at fixed $r_0'<0$, since $w\sim O(h_0)$ according to Eq. (84). These Goldstone divergences, however, are canceled exactly by the four-loop term $\mathring{\Gamma}_4$ of the Helmholtz free energy $\mathring{\Gamma}$ obtained within step (ii) of our calculation in the following section.

B. Contribution of $O(u_0^3)$ from $\mathring{\Gamma}_4$

Substituting the zeroth-order term $M_0^2=(4u_0)^{-1}(-r_0'+\mathring{\chi}_T^{-1})$ into $\mathring{\Gamma}_4(r_0',u_0,M_0)$ we obtain the four-loop contribution at small w

$$\mathring{\Gamma}_4=\frac{u_0^3}{(4\pi)^4}\Bigg\{\kappa(n)+(n+2)\left(16n^2-48n+128-\frac{4}{3}n(n+8)\pi^2\right)\ln\frac{-2r_0'}{(24u_0)^2}+189\pi^2-108-1008\ln 3+7776\ln 2-972\mathrm{Li}_2\left(-\frac{1}{3}\right)-3888c_1+(n-1)\left(-\frac{36}{w^{3/2}}+\frac{12}{w^{1/2}}\left[n^2+41-8(n-1)\ln 3+4(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\right]+\frac{8}{3}n^3-32n^2\ln 3+\left[460-\frac{59}{3}\pi^2+96\ln 3\right]n-\frac{44\,684}{45}-\frac{3436}{9}\pi^2-\frac{28\,688}{5}\ln 2+3338(\ln 3)^2+\frac{16\,412}{3}\mathrm{Li}_2\left(\frac{1}{3}\right)-48c_2\right)\Bigg\}+O(w^{1/2},w^{1/2}\ln w,u_0^4,u_0^4\ln u_0), \tag{90}$$

where w is defined in Eq. (84) and

$$\kappa(n)=144\left[-20\ln 5+9\mathrm{Li}_2\left(-\frac{1}{4}\right)+9\mathrm{Li}_2\left(-\frac{2}{3}\right)+\frac{9}{2}\ln\left(\frac{3}{4}\right)^2+216J_{2,1}^{(1)}-162J_{2,2}^{(1)}-18c_4-36J_{1,1}^{(1)}+108A_1-324D_1-36E_1+216F_1-36G_1-216H_1\right]+(n-1)\Bigg\{\left(28\zeta(3)-2\pi^2+\frac{8}{3}\pi^2\ln 2\right)n^2+\Bigg(\frac{617}{2}\zeta(3)-96c_3-\frac{115}{6}\pi^2\ln 3+\frac{88}{3}\pi^2\ln 2+39\left[\frac{(\ln 3)^3}{6}+\mathrm{polylog}\left(3,-\frac{1}{3}\right)-2\,\mathrm{polylog}\left(3,\frac{1}{3}\right)\right]\Bigg)n-\frac{633}{2}\zeta(3)+3296\ln 3+360(\ln 2)^2-\frac{14\,608}{3}\ln 2\ln 3-\frac{165}{2}(\ln 3)^3-\frac{53}{6}\pi^2\ln 3+\frac{536}{3}\pi^2\ln 2-63\,\mathrm{polylog}\left(3,-\frac{1}{3}\right)+558\,\mathrm{polylog}\left(3,\frac{1}{3}\right)-1152J_{1,1}^{(0)}+3456(J_{1,2}^{(0)}+J_{2,1}^{(0)})-8(648J_{2,2}^{(0)}+52c_3+X_{410}-2X_{420}+4X_{430})\Bigg\}. \tag{91}$$

We see that the same terms $\sim w^{-3/2}$ and $w^{-1/2}$ appear in Eq. (90) as in Eq. (88), but with an opposite sign. The sum of Eqs. (88) and (90) yields the Gibbs free energy for $r_0'<0$ up to four-loop order for $d=3$

$$\mathring{\mathcal{F}}_-(r_0',u_0)=\mathring{\mathcal{F}}_-(r_0',u_0)^{\text{three-loop}}+\frac{u_0^3}{(4\pi)^4}\left\{\Lambda_4(n)-16(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\left[(n+2)\ln\frac{-2r_0'}{(24u_0)^2}+2n-8-4(n-1)\ln 3+\frac{\pi^2}{12}n(n+8)\right]\right\}+O(u_0^4,u_0^4\ln u_0), \tag{92}$$

$$\Lambda_4(n)=\kappa(n)+648+378\pi^2+1152\ln 24+1296\mathrm{Li}_2\left(-\frac{1}{3}\right)-2592c_1+(n-1)\left[\left(8-\frac{14}{3}\pi^2+64(1-\ln 3)\ln 3\right)n-4672\ln 2+\frac{2024}{9}-\frac{2734}{9}\pi^2+\frac{15\,440}{3}\mathrm{Li}_2\left(\frac{1}{3}\right)+3240(\ln 3)^2-96c_2\right] \tag{93}$$

$$=32.16\,n^3+205.9\,n^2+404.9\,n+673.5. \tag{94}$$

The terms of $O(w^{1/2})$ and $O(w^{1/2}\ln w)$ in Eqs. (88) and (90) do not contribute to the free energy $\mathring{\mathcal{F}}_-(r_0',u_0)$ and to the specific heat $\partial^2\mathring{\mathcal{F}}_-(r_0',u_0)/(\partial r_0')^2$ on the coexistence curve, because these terms vanish in the limit $h_0\to 0$ $(w\to 0)$. Therefore it was sufficient to evaluate most of the diagrams in Fig. 1 directly at $r_{0T}=0$ $(w=0)$.

It is straightforward to derive from the Gibbs free energy $\mathring{\mathcal{F}}_\pm(r_0',u_0)$ the bare vertex functions $\mathring{\Gamma}_\pm^{(1,0)}(r_0',u_0)=\partial\mathring{\mathcal{F}}_\pm(r_0',u_0)/\partial r_0'$ and $\mathring{\Gamma}_\pm^{(2,0)}(r_0',u_0)=\partial^2\mathring{\mathcal{F}}_\pm(r_0',u_0)/(\partial r_0')^2$. The bare perturbation series of $\mathring{\Gamma}_\pm^{(2,0)}(r_0',u_0)$ up to four-loop order is given in Appendix C.

C. $\mathring{\mathcal{F}}_{\pm}$ as a function of the correlation lengths

Equations (73) and (88)–(92) contain logarithms of the coupling u_0 as expected because of the nonanalytic u_0 dependence of δr_0, Eq. (23). Apart from one special four-loop logarithmic term in Eq. (92) to be discussed below, these logarithms can be absorbed by expressing r_0' in terms of the correlation lengths ξ_+ and ξ_- for $r_0'>0$ and $r_0'<0$, respectively [30,31]. Substituting Eqs. (58)–(62) and (66)–(68) into Eqs. (73) and (92) yields the Gibbs free energy up to four-loop order as a function of $\xi_\pm$ in $d=3$ dimensions

$$\mathring{\mathcal{F}}_{\pm}(r_0'(\xi_\pm,u_0),u_0) = \xi_\pm^{-3}\left\{ \sum_{m=0}^{4} a_{\pm m}^{(\Gamma)}(u_0\xi_\pm)^{m-1} + \frac{n(n+2)(n+8)}{96\pi^2} \times (u_0\xi_\pm)^3 \ln(u_0\xi_\pm) + O(u_0^4\xi_\pm^4)\right\}. \quad (95)$$

Above T_c, the analytic result is

$$a_{+0}^{(\Gamma)}=0, \quad (96)$$

$$a_{+1}^{(\Gamma)}=-\frac{n}{12\pi}, \quad (97)$$

$$a_{+2}^{(\Gamma)}=-\frac{n(n+2)}{16\pi^2}, \quad (98)$$

$$a_{+3}^{(\Gamma)}=\frac{n(n+2)}{8\pi^3}\left[\frac{53}{27}-2\ln\frac{4}{3}\right], \quad (99)$$

with the four-loop coefficient for general n

$$a_{+4}^{(\Gamma)}=-\frac{n(n+2)}{3\pi^4}\left[\frac{n}{16}+\frac{11}{24}-(n+8)\left(\frac{\pi^2}{12}+\mathrm{Li}_2\left(-\frac{1}{3}\right)\right) + \frac{43n+182}{72}\ln\frac{4}{3}-\frac{\pi^2}{32}(n+8)\ln 24\right]. \quad (100)$$

Equations (96)–(99) agree with the corrected form of the previous three-loop result for $\mathring{\mathcal{F}}_{+}(r_0'(\xi_+,u_0),u_0)$ [36]. Below T_c, the analytic result is

$$a_{-0}^{(\Gamma)}=-\frac{1}{64}, \quad (101)$$

$$a_{-1}^{(\Gamma)}=-\frac{1}{96\pi}(3n+14), \quad (102)$$

$$a_{-2}^{(\Gamma)}=-\frac{1}{3456\pi^2}[54n^2-737n-394+432(n-1)\ln 3], \quad (103)$$

$$a_{-3}^{(\Gamma)}=\frac{1}{3456\pi^3}\left[179n^2-3875n-10\,626-116\,64c_1-1134 \times(n-1)\left[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2+\frac{8}{21}c_2\right]+16\,(43n^2+547n+1219)\ln 2-8\,(97n^2+538n+1174)\ln 3 +(33n^2-183n-903)\,\pi^2-36\,(16n^2+160n+175)\mathrm{Li}_2\left(-\frac{1}{3}\right)\right], \quad (104)$$

with the four-loop coefficient for general n

$$a_{-4}^{(\Gamma)}=\pi^{-4}\left\{\frac{32\,772\,763}{331\,776}-\frac{645}{256}\pi^2+\frac{81}{1280}\pi^4+\frac{2673}{256}\zeta(3)+\frac{301}{24}\ln\frac{4}{3}+\frac{179}{192}\ln\frac{5}{3}-\frac{2811}{64}\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{9}{32}\left[\pi^2\ln 24-(\ln 2)^2 -2\mathrm{Li}_2\left(\frac{1}{6}\right)+2\mathrm{Li}_2\left(\frac{1}{3}\right)\right]+\frac{21}{8}\left[\mathrm{Li}_2\left(-\frac{2}{3}\right)+\mathrm{Li}_2\left(-\frac{1}{4}\right)+\frac{1}{2}\left(\ln\frac{3}{4}\right)^2\right]+\frac{27}{4}J_{1,1}^{(1)}+108J_{2,1}^{(1)}+27J_{3,1}^{(1)}-\frac{729}{8}J_{2,2}^{(1)}-\frac{243}{16}c_1 +\frac{81}{8}c_4+\frac{81}{4}[3A_1-9D_1+6F_1-G_1-6H_1]-\frac{27}{4}E_1'+27E_1''\right\}+\frac{n-1}{\pi^4}\left\{\left(\frac{5129}{110\,592}-\frac{151}{6912}\pi^2+\frac{37}{256}\zeta(3)+\frac{25}{144}\ln\frac{4}{3} -\frac{7}{18}\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{\pi^2}{96}\ln 48+\frac{c_4}{4}\right)n^2+\left(\frac{9\,366\,787}{2\,985\,984}-\frac{323}{864}\pi^2+\frac{\pi^4}{256}+\frac{971}{512}\zeta(3)+\frac{89}{48}\ln 2-\frac{665}{1728}\ln 3+\frac{797}{576}\ln 5 -\frac{169}{576}\mathrm{Li}_2\left(\frac{1}{3}\right)-\frac{563}{108}\mathrm{Li}_2\left(-\frac{1}{3}\right)-\frac{173}{432}\left[\mathrm{Li}_2\left(-\frac{2}{3}\right)+\mathrm{Li}_2\left(-\frac{1}{4}\right)\right]-\frac{5}{144}\mathrm{Li}_2\left(\frac{1}{6}\right)-\frac{2123}{3456}(\ln 3)^2-\frac{707}{864}(\ln 2)^2+\frac{173}{216}(\ln 3) \times(\ln 2)+\frac{13}{512}(\ln 3)^3+\frac{61}{1536}\pi^2\ln 3+\frac{11}{24}\pi^2\ln 2+\frac{39}{256}\left[\mathrm{polylog}\left(3,-\frac{1}{3}\right)-2\,\mathrm{polylog}\left(3,\frac{1}{3}\right)\right]-\frac{1}{16}c_2-\frac{3}{8}c_3+\frac{9}{4}c_4 +\frac{5}{12}[4J_{1,1}^{(1)}-2J_{2,1}^{(1)}+4J_{3,1}^{(1)}+3E_1-E_1'+4E_1'']\right)n+\frac{116\,414\,987}{2\,985\,984}-\frac{21\,937}{6912}\pi^2+\frac{37}{1280}\pi^4+\frac{1785}{512}\zeta(3)-\frac{869}{72}\ln 2$$

$$+\frac{10\,901}{1728}\ln 3+\frac{437}{64}\ln 5-\frac{41\,563}{1728}\mathrm{Li}_2\left(-\frac{1}{3}\right)-\frac{697}{432}\left[\mathrm{Li}_2\left(-\frac{1}{4}\right)+\mathrm{Li}_2\left(-\frac{2}{3}\right)\right]+\frac{5675}{288}\mathrm{Li}_2\left(\frac{1}{3}\right)-\frac{37}{144}\mathrm{Li}_2\left(\frac{1}{6}\right)$$
$$-\frac{421}{216}(\ln 2)^2-\frac{6823}{432}(\ln 2)(\ln 3)+\frac{19\,909}{1728}(\ln 3)^2-\frac{165}{512}(\ln 3)^3+\frac{379}{1536}\pi^2\ln 3+\frac{37}{24}\pi^2\ln 2+\frac{279}{128}\mathrm{polylog}\left(3,\frac{1}{3}\right)$$
$$-\frac{63}{256}\mathrm{polylog}\left(3,-\frac{1}{3}\right)-\frac{9}{2}J_{1,1}^{(0)}+\frac{27}{2}[J_{1,2}^{(0)}+J_{2,1}^{(0)}]-\frac{81}{4}J_{2,2}^{(0)}+\frac{37}{12}[4J_{1,1}^{(1)}-2J_{2,1}^{(1)}+4J_{3,1}^{(1)}+3E_1-E_1'+4E_1'']-\frac{27}{16}c_1$$
$$-\frac{1}{2}c_2-\frac{13}{8}c_3+\frac{41}{4}c_4-\frac{1}{32}X_{410}+\frac{1}{16}X_{420}+\frac{1}{8}X_{430}\Bigg\}. \quad (105)$$

The coefficients (101)–(104) agree with the previous three-loop result for $\mathring{\mathcal{F}}_-(r_0'(\xi_-,u_0),u_0)$ in Eqs. (35)–(39) of Ref. [36], where $a_{-3}^{(\Gamma)}$ was given in a slightly different form, see our Eqs. (A54) and (A55). In numerical form, Eq. (105) reads (up to five digits)

$$a_{-4}^{(\Gamma)}=0.006\,209\,8\,n^3+0.099\,061\,n^2+0.490\,03\,n+0.650\,73. \quad (106)$$

The numerical value of $a_{-4}^{(\Gamma)}$ for $n=1$ agrees with that given in Table 2 of Ref. [33] for the first five digits. The remaining logarithmic four-loop contribution of $O(u_0^3\ln(u_0\xi_\pm))$ in Eq. (95) corresponds to that in Eq. (3.15) of Ref. [33] for $n=1$. This logarithm is not caused by the mass shift δr_0 but it originates from those diagrams (of type B in Fig. 1) that yield the additive $d=3$ pole term P_4, Eq. (28). For the special case $n=1$, the numerical value of this logarithmic term in Eq. (95) agrees with the coefficient $\tilde{a}_{-4}^{(\Gamma)}$ in Eq. (3.15) of Ref. [33].

It is straightforward to derive the vertex functions $\mathring{\Gamma}_\pm^{(1,0)}(r_0'(\xi_\pm,u_0),u_0)$ and $\mathring{\Gamma}_\pm^{(2,0)}(r_0'(\xi_\pm,u_0),u_0)$ from $\mathring{\mathcal{F}}_\pm$. The perturbation series of the vertex function $\mathring{\Gamma}_\pm^{(2,0)}(r_0'(\xi_\pm,u_0),u_0)$ up to four-loop order is given in Appendix C.

V. RENORMALIZATION AND AMPLITUDE FUNCTIONS

The shift of the temperature variable r_0 by δr_0, Eq. (23), and the subtraction $\delta\Gamma_0$, Eq. (30), are sufficient to make the Gibbs free energy $\mathring{\mathcal{F}}_\pm(r_0'(\xi_\pm,u_0),u_0)$, Eq. (95), and the vertex functions $\mathring{\Gamma}_\pm^{(L,0)}(r_0'(\xi_\pm,u_0),u_0)$ finite in three dimensions at infinite cutoff as long as $\xi_\pm$ is finite. In the critical limit $\xi_\pm\to\infty$ at fixed u_0, however, the bare perturbative form of $\mathring{\mathcal{F}}_\pm$, Eq. (95), is formally divergent. In order to make this perturbation series applicable near T_c two steps need to be performed: (i) the series has to be mapped from the critical to the noncritical region, (ii) the mapped series has to be resummed. Step (i) is achieved by turning to the renormalized theory as defined below and by introducing the renormalization scale μ that can be varied via the renormalization-group equation (RGE) [18,56]. Step (ii) will be performed in Sec. VI by means of a variational approach.

In the following, we define renormalized vertex functions in $2<d<4$ dimensions and calculate their amplitude functions in $d=3$ dimensions up to four-loop order. Our approach is a combination of the minimal subtraction scheme [19] and of massive field theory at fixed dimension [17] $d<4$, without using the $\varepsilon=4-d$ expansion, as introduced in Refs. [29–31] and further discussed in Refs. [34,36]. A significant advantage of this approach is the fact that the additive renormalization $A(u,\varepsilon)$ and the multiplicative renormalizations $Z_r(u,\varepsilon),Z_\varphi(u,\varepsilon)$, and $Z_u(u,\varepsilon)$ are the same above and below T_c.

The bare Gibbs free energy as a function of the correlation lengths ξ_+ and ξ_- in d dimensions will be denoted by $\mathring{\mathcal{F}}_\pm(\xi_\pm,u_0,d)$ which for $d=3$ is identical with $\mathring{\mathcal{F}}_\pm(r_0'(\xi_\pm,u_0),u_0)$ of Eq. (95). Correspondingly we use the notation $\mathring{\Gamma}_\pm^{(L,0)}(\xi_\pm,u_0,d)$ for the bare vertex functions in d dimensions as functions of ξ_+ and ξ_-. The renormalized quantities are introduced as

$$r=Z_r^{-1}(r_0-r_{0c}), \quad (107)$$

$$u=\mu^{-\varepsilon}A_dZ_u^{-1}Z_\varphi^2u_0, \quad (108)$$

$$\boldsymbol{\varphi}=Z_\varphi^{-1/2}\boldsymbol{\varphi}_0, \quad (109)$$

$$\mathcal{F}_\pm(\xi_\pm,u,\mu,d)=\mathring{\mathcal{F}}_\pm(\xi_\pm,\mu^\varepsilon Z_uZ_\varphi^{-2}A_d^{-1}u,d)-\frac{1}{8}\mu^{-\varepsilon}r^2A_dA(u,\varepsilon), \quad (110)$$

$$\Gamma_\pm^{(1,0)}(\xi_\pm,u,\mu,d)=Z_r\mathring{\Gamma}_\pm^{(1,0)}(\xi_\pm,\mu^\varepsilon Z_uZ_\varphi^{-2}A_d^{-1}u,d)-\frac{1}{4}\mu^{-\varepsilon}r\,A_dA(u,\varepsilon), \quad (111)$$

$$\Gamma_\pm^{(2,0)}(\xi_\pm,u,\mu,d)=Z_r^2\mathring{\Gamma}_\pm^{(2,0)}(\xi_\pm,\mu^\varepsilon Z_uZ_\varphi^{-2}A_d^{-1}u,d)-\frac{1}{4}\mu^{-\varepsilon}A_dA(u,\varepsilon), \quad (112)$$

where

$$A_d = \frac{\Gamma(3-d/2)}{2^{d-2}\pi^{d/2}(d-2)} \tag{113}$$

is an appropriate geometric factor [29,30]. The analytic form of the renormalization constants $Z_r(u,\varepsilon)$, $Z_u(u,\varepsilon)$, $Z_\varphi(u,\varepsilon)$, and $A(u,\varepsilon)$ is given in Eqs. (2.13), (2.16)–(2.19), and (B1)–(B18) of Ref. [35] for general n up to five-loop order.

A. Amplitude functions in $2<d<4$ dimensions

Dimensionless amplitude functions of the renormalized quantities can be defined for $2<d<4$ as

$$f_\pm^{(0,0)}(\mu\xi_\pm,u,d) = \mu^{-d}A_d^{-1}\mathcal{F}_\pm(\xi_\pm,u,\mu,d), \tag{114}$$

$$f_\pm^{(1,0)}(\mu\xi_\pm,u,d) = -2\mu^{2-d}A_d^{-1}\Gamma_\pm^{(1,0)}(\xi_\pm,u,\mu,d), \tag{115}$$

$$F_\pm(\mu\xi_\pm,u,d) = -4\mu^{\varepsilon}A_d^{-1}\Gamma_\pm^{(2,0)}(\xi_\pm,u,\mu,d), \tag{116}$$

$$Q_+(\mu\xi_+,u,d) = r/\mu^2 = \mu^{-2}Z_r^{-1}\mathring{h}(\xi_+,\mu^{\varepsilon}Z_uZ_\varphi^{-2}A_d^{-1}u,d), \tag{117}$$

$$P_+(\mu\xi_+,u,d) = (\partial r/\partial\xi_+^{-2})_{u_0} = Z_r^{-1}\mathring{P}_+(\mu^{\varepsilon}\xi_+^{\varepsilon}Z_uZ_\varphi^{-2}A_d^{-1}u,d). \tag{118}$$

These functions remain finite also in the limit $d\to4$ (at finite $\xi_\pm$) [30,31]. We recall that the amplitude functions depend on the choice of the geometric factor A_d. Our choice, Eq. (113), minimizes the explicit dimensional dependence of the lowest-order coefficients of the amplitude functions and thereby simplifies their analytic form at low order [29–31]. It is expected that the convergence properties of the perturbation series of the amplitude functions is significantly affected by the choice of A_d as will be discussed elsewhere.

From the μ independence of the bare quantities we derive the RGEs for the amplitude functions $f_\pm^{(0,0)}$, $f_\pm^{(1,0)}$, and $F_\pm$

$$0 = (\mu\,\partial_\mu + \beta_u\partial_u + d)[f_\pm^{(0,0)}(\mu\xi_\pm,u,d) + \tfrac{1}{8}q_\pm(\mu\xi_\pm,u,d)^2A(u,\varepsilon)], \tag{119}$$

$$0 = (\mu\,\partial_\mu + \beta_u\partial_u + d - 2 + \zeta_r)[f_\pm^{(1,0)}(\mu\xi_\pm,u,d) - \tfrac{1}{2}q_\pm(\mu\xi_\pm,u,d)A(u,\varepsilon)], \tag{120}$$

$$4B(u) = (\mu\partial_\mu + \beta_u\partial_u + 2\zeta_r - \varepsilon)F_\pm(\mu\xi_\pm,u,d), \tag{121}$$

where $B(u)$ is defined by

$$4B(u) = [2\zeta_r - \varepsilon]A(u,\varepsilon) + \beta_u(u,\varepsilon)\frac{\partial A(u,\varepsilon)}{\partial u}. \tag{122}$$

The expansion coefficients of the standard field-theoretic functions $\beta_u(u,\varepsilon),\zeta_r(u)$, and $B(u)$ can be obtained from the renormalization constants in Ref. [35] up to five-loop order. To simplify the notation, we have used the functions $q_+(\mu\xi_+,u,d) = r/\mu^2$ for $r>0$ and $q_-(\mu\xi_-,u,d) = r/\mu^2$ for $r<0$, which are related to the amplitude functions $Q_\pm$ of Refs. [30,31] according to

$$q_+(\mu\xi_+,u,d) = Q_+(\mu\xi_+,u,d), \tag{123}$$

$$q_-(\mu\xi_-,u,d) = -\tfrac{1}{2}Q_-(\mu\xi_-,u,d). \tag{124}$$

Integration of the RGE yields

$$f_\pm^{(0,0)}(\mu\xi_\pm,u,d) = \left\{ f_\pm^{(0,0)}(1,u(l_\pm),d) + \frac{1}{2}q_\pm(1,u(l_\pm),d)^2\int_1^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\} l_\pm^d, \tag{125}$$

$$f_\pm^{(1,0)}(\mu\xi_\pm,u,d) = \left\{ f_\pm^{(1,0)}(1,u(l_\pm),d) - 2q_\pm(1,u(l_\pm),d)\int_1^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\} \times\exp\left(\int_1^{l_\pm}(\zeta_r+d-2)\frac{dl'''}{l'''}\right), \tag{126}$$

$$F_\pm(\mu\xi_\pm,u,d) = \left\{ F_\pm(1,u(l_\pm),d) - 4\int_1^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\}\exp\left(\int_1^{l_\pm}(2\zeta_r-\varepsilon)\frac{dl'''}{l'''}\right), \tag{127}$$

with $l_\pm = (\mu\xi_\pm)^{-1}$, where the effective coupling $u(l_\pm)$ satisfies

$$l_\pm\frac{du(l_\pm)}{dl_\pm} = \beta_u(u(l_\pm),\varepsilon). \tag{128}$$

Furthermore, we have [30,31]

$$q_\pm(\mu\xi_\pm,u,d) = q_\pm(1,u(l_\pm),d)\exp\left(\int_1^{l_\pm}(2-\zeta_r)\frac{dl'}{l'}\right), \tag{129}$$

$$P_{\pm}(\mu\xi_{\pm},u,d)=P_{\pm}(1,u(l_{\pm}),d)\exp\left(\int_{l_{\pm}}^{1}\zeta_r\frac{dl'}{l'}\right). \tag{130}$$

At $\mu\xi_{\pm}=1$ these functions are related by

$$q_{-}(1,u,d)=q_{+}(1,u,d)-\tfrac{3}{2}, \tag{131}$$

$$P_{-}(1,u,d)=-\tfrac{3}{4}[2-\zeta_r(u)]+P_{+}(1,u,d). \tag{132}$$

Substituting Eqs. (125) and (127) into Eqs. (110), (112), (114), and (116), respectively, and using Eqs. (C1) and (C2) of Appendix C, we arrive at the following representation of the free energy and of the specific heat

$$\begin{aligned}\mathring{\mathcal{F}}_{\pm}(\xi_{\pm},u_0,d)=A_d\,\xi_{\pm}^{-d}\Bigg\{&f_{\pm}^{(0,0)}(1,u(l_{\pm}),d)\\&+\frac{1}{2}q_{\pm}(1,u(l_{\pm}),d)^2\int_1^{l_{\pm}}B(u(l'))\\&\times\left[\exp\left(\int_{l_{\pm}}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\Bigg\}\\&+\frac{1}{8}A_d\mu^{-\varepsilon}A(u,\varepsilon)r^2\end{aligned} \tag{133}$$

and

$$\begin{aligned}C^{\pm}(t)=C_B+\mu^{-\varepsilon}A_dZ_r(u,\varepsilon)^{-2}K_{\pm}(u(l_{\pm}),\varepsilon)\\\times\exp\left(\int_1^{l_{\pm}}[2\zeta_r(u(l'))-\varepsilon]\frac{dl'}{l'}\right),\end{aligned} \tag{134}$$

with

$$K_{\pm}(u,\varepsilon)=F_{\pm}(1,u,d)-A(u,\varepsilon). \tag{135}$$

From Eqs. (12)–(16), (31) and (72)–(74) it is clear that the quantity $\mathring{\mathcal{F}}_{\pm}$, Eq. (133), is to be identified with the singular part of the free energy of Sec. I including the regular term $-\frac{1}{2}B_{cr}t^2$,

$$\mathring{\mathcal{F}}_{\pm}(\xi_{\pm}(t),u_0,d)=f_s^{\pm}(t)-\tfrac{1}{2}B_{cr}t^2, \tag{136}$$

apart from cutoff dependent contributions. Equations (133) and (134) will be evaluated asymptotically ($\xi_{\pm}\to\infty$) in Sec. VI and Appendix E.

Equations (125)–(130) provide the mapping of the amplitude functions from the critical region $\mu\xi_{\pm}\gg 1$ to the noncritical value $\mu\xi_{\pm}=1$. At this value the amplitude functions are related by the differential equations

$$q_{+}(1,u,d)=[2-\zeta_r(u)]^{-1}\left[2P_{+}(1,u,d)-\beta_u(u,\varepsilon)\frac{\partial q_{+}(1,u,d)}{\partial u}\right], \tag{137}$$

$$f_{\pm}^{(1,0)}(1,u,d)=-P_{\pm}(1,u,d)^{-1}\left[\left(d+\beta_u(u,\varepsilon)\frac{\partial}{\partial u}\right)f_{\pm}^{(0,0)}(1,u,d)+\frac{1}{2}q_{\pm}(1,u,d)^2B(u)\right], \tag{138}$$

$$F_{\pm}(1,u,d)=P_{\pm}(1,u,d)^{-1}\left[\left(d-2+\zeta_r(u)+\beta_u(u,\varepsilon)\frac{\partial}{\partial u}\right)f_{\pm}^{(1,0)}(1,u,d)-2q_{\pm}(1,u,d)B(u)\right] \tag{139}$$

$$=P_{\pm}(1,u,d)^{-1}\left(d-2+\zeta_r(u)+\beta_u(u,\varepsilon)\frac{\partial}{\partial u}\right)\left[f_{\pm}^{(1,0)}(1,u,d)-\frac{1}{2}q_{\pm}(1,u,d)A(u,\varepsilon)\right]+A(u,\varepsilon). \tag{140}$$

Unlike the functions $P_{\pm}(1,u,d)$ and $F_{\pm}(1,u,d)$, which have expansions in integer powers of u, the functions $q_{\pm}(1,u,d), f_{\pm}^{(0,0)}(1,u,d)$, and $f_{\pm}^{(1,0)}(1,u,d)$ are not expandable in integer powers of u, which is a consequence of the nonanalytic u_0 dependence of $r_{0c}(u_0,\varepsilon)$, Eq. (20) [30,31]. Therefore, the perturbative expansions of these functions are not Borel resummable. They can be expressed, however, in terms of the Borel-resummable functions $P_{\pm}(1,u,d)\equiv P_{\pm}(u)$ and $F_{\pm}(1,u,d)\equiv F_{\pm}(u)$ via the integral representations

$$q_{+}(1,u,d)=\int_{u^*}^{u}du'\frac{2P_{+}(u')}{\beta_u(u',\varepsilon)}\exp\left(\int_u^{u'}du''\frac{2-\zeta_r(u'')}{\beta_u(u'',\varepsilon)}\right), \tag{141}$$

$$\begin{aligned}f_{\pm}^{(1,0)}(1,u,d)=\int_{u^*}^{u}&\frac{P_{\pm}(u')F_{\pm}(u')+2q_{\pm}(1,u',d)B(u')}{\beta_u(u',\varepsilon)}\\&\times\left[\exp\left(\int_u^{u'}\frac{d+\zeta_r(u'')-2}{\beta_u(u'',\varepsilon)}du''\right)\right]du',\end{aligned} \tag{142}$$

$$f_{\pm}^{(0,0)}(1,u,d)=\int_u^{u^*}du'\left(\frac{P_{\pm}(u')f_{\pm}^{(1,0)}(1,u',d)}{\beta_u(u',\varepsilon)}+\frac{q_{\pm}(1,u',d)^2B(u')}{2\beta_u(u',\varepsilon)}\right)\times\exp\left(\int_u^{u'}du''\frac{d}{\beta_u(u'',\varepsilon)}\right). \quad (143)$$

Equations (141)–(143) follow directly from Eqs. (137)–(139), as can be verified by differentiating the former equations. Note that the right-hand sides of Eqs. (141)–(143) have finite limits for $u\to u^*$, see Eqs. (169)–(171) below.

B. Four-loop amplitude functions in three dimensions

In three dimensions we have from Eqs. (110), (114), (117), (123), and (124),

$$f_{\pm}^{(0,0)}(1,u,3)=4\pi\mu^{-3}\mathring{\mathcal{F}}_{\pm}(\mu^{-1},4\pi\mu Z_u Z_{\varphi}^{-2}u,3)-\frac{1}{8}q_{\pm}(1,u,3)^2A(u,1). \quad (144)$$

The four-loop expressions of the functions $q_{\pm}(1,u,3)$ follow from the four-loop result for $\mathring{h}(\xi_+,u_0,3)$ and are given in Appendix B. These expressions contain logarithmic u dependences. Substituting Eqs. (95)–(105) into Eq. (144) and using Eq. (138), we obtain the four-loop expressions of $f_{\pm}^{(0,0)}(1,u,3)$ and $f_{\pm}^{(1,0)}(1,u,3)$. They are given in Appendix D. Because of the functions $q_{\pm}$ in Eqs. (138) and (144) the four-loop expansions of the functions $f_{\pm}^{(0,0)}(1,u,3)$ and $f_{\pm}^{(1,0)}(1,u,3)$ contain logarithmic terms. In addition, $f_{\pm}^{(0,0)}$ has a special logarithmic four-loop term arising from the diagrams that cause the $d=3$ pole term, Eq. (28).

The amplitude functions $F_{\pm}$ can be derived from $f_{\pm}^{(1,0)}$ according to Eq. (140). Unlike $f_{\pm}^{(0,0)}$ and $f_{\pm}^{(1,0)}$, the functions $F_+(1,u,3)$ and $uF_-(1,u,3)$ can be expanded around $u=0$ in integer powers of u, i.e., they do not contain logarithmic u dependences, since $f_{\pm}^{(1,0)}-\frac{1}{2}q_{\pm}A$ is free of logarithms (see Appendix D). In three dimensions, we obtain the power series

$$F_{\pm}(u)\equiv F_{\pm}(1,u,3)=\sum_{m=0}^{\infty}c_{Fm}^{\pm}u^{m-1}. \quad (145)$$

The coefficients read up to four-loop order

$$c_{F0}^{+}=0, \quad (146)$$

$$c_{F1}^{+}=-n, \quad (147)$$

$$c_{F2}^{+}=-2n(n+2), \quad (148)$$

$$c_{F3}^{+}=-4n(n+2)\left[n-\frac{7}{27}+4\ln\frac{4}{3}\right], \quad (149)$$

$$c_{F4}^{+}=8n(n+2)\left[-n^2+\frac{233}{27}n+\frac{1888}{27}+(n+8)\left(\frac{2\pi^2}{9}+\frac{32}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)-4\zeta(3)\right)+\frac{8}{27}(89n+550)\ln\frac{3}{4}\right] \quad (150)$$

and

$$c_{F0}^{-}=\frac{1}{2}, \quad (151)$$

$$c_{F1}^{-}=-4, \quad (152)$$

$$c_{F2}^{-}=8(10-n), \quad (153)$$

$$c_{F3}^{-}=-\frac{1}{27}(1080n^2+3464n+31\,120)-128\,(5n+22)\zeta(3)-864c_1-32(n-1)c_2+(6n^2+22n-10)\pi^2-84(n-1)\left[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\right]+216\mathrm{Li}_2\left(-\frac{1}{3}\right)-32(4n+17)\ln 3+\frac{32}{3}(31n+95)\ln 2, \quad (154)$$

$$c_{F4}^{-}=-\frac{1}{27}(7344n^3+124\,432n^2-108\,512n-1\,008\,512)-\frac{64}{15}(n+5)(5n+22)\pi^4-128(n^3-5n^2-168n-692)\zeta(3)+1280(2n^2+55n+186)\zeta(5)-8(5n+34)\left\{216c_1+8(n-1)c_2+21(n-1)\times\left[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\right]\right\}-\frac{16}{3}(128n^2+875n-706)\times\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{4\pi^2}{9}(87n^3+805n^2+1093n-3578)-\frac{64}{27}(280n^2+6607n+17\,926)\ln 3+\frac{64}{27}(1739n^2+15\,905n+31\,982)\ln 2 \quad (155)$$

above and below T_c, respectively. The terms up to $m=3$ are identical with the previous three-loop results [36], except that here we have written c_{F3}^{-} in a more compact form using the relations (A54) and (A55) of Appendix A.

Equations (145)–(155) are the basis for future Borel resummations and for the variational calculations in Sec. VII. These calculations can be performed for general $u>0$ (not only at the fixed point u^*), which is of relevance for a nonlinear RG analysis of the nonasymptotic critical behavior [43,44].

In order to check the correctness of Eqs. (144)–(155), we present in Appendix C an alternative procedure for deriving $F_\pm(1,u,3)$ by starting directly from the perturbation series of $\mathring{\Gamma}_\pm^{(2,0)}(r_0',u_0)$. The resulting coefficients $c_{Fm}^\pm$ agree with Eqs. (146)–(155).

The amplitude function $P_+(\mu\xi_+,u,d)=Z_r^{-1}(\partial r_0'/\partial\xi_+^{-2})_{u_0}$ above T_c can be derived from Eqs. (56)–(61) for $d=3$. The resulting series at $\mu\xi_+=1$,

$$P_+(1,u,3)=\sum_{m=0}^{\infty} c_{Pm}u^m, \tag{156}$$

is free of logarithms. The coefficients read in analytic form for general n up to four-loop order

$$c_{P0}=1, \tag{157}$$

$$c_{P1}=-2(n+2), \tag{158}$$

$$c_{P2}=4(n+2), \tag{159}$$

$$c_{P3}=-(n+2)\left[56n+\frac{4576}{9}-64\pi^3\lambda_3\right], \tag{160}$$

$$c_{P4}=32(n+2)\left[\frac{\pi^4}{15}(5n+22)+3(n^2-10n-36)\zeta(3)-\frac{4117}{432}n^2-\frac{535967}{5832}n-\frac{1441439}{2916}+\tau(n)\right]. \tag{161}$$

with λ_3 given by Eq. (60) and

$$\tau(n)=\frac{32}{3}(5n+22)\left(-4J_{1,1}^{(1)}+2J_{2,1}^{(1)}-4J_{3,1}^{(1)}+\frac{1}{12}\left[\mathrm{Li}_2\left(\frac{1}{6}\right)-\mathrm{Li}_2\left(\frac{1}{3}\right)+\frac{1}{2}(\ln 2)^2\right]-3E_1+E_1'-4E_1''\right)+\frac{8}{27}(173n+178)\times\left[\frac{1}{2}\left(\ln\frac{3}{4}\right)^2+\mathrm{Li}_2\left(-\frac{1}{4}\right)+\mathrm{Li}_2\left(-\frac{2}{3}\right)\right]-32(n^2+6n+20)c_4+\frac{\pi^2}{27}(89n^2+1472n+5324)+\frac{32}{27}(42n^2+527n+2050)\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{1}{27}(600n^2+2008n+8528)\ln\frac{3}{4}-\frac{2}{9}(797n+1542)\ln\frac{5}{3}. \tag{162}$$

The numerical values of these coefficients for $n=1,2,3$ agree with those given in Table 4 of Ref. [32]. Below T_c, we have

$$P_-(1,u,3)\equiv P_+(1,u,3)+\frac{3}{4}[\zeta_r(u)-2]=\sum_{m=0}^{4} c_{Pm}^- u^m. \tag{163}$$

Equations (157)–(163) yield

$$c_{P0}^-=-\frac{1}{2}, \tag{164}$$

$$c_{P1}^-=(n+2), \tag{165}$$

$$c_{P2}^-=-26(n+2), \tag{166}$$

$$c_{P3}^-=(n+2)\left[64\pi^3\lambda_3+124n+\frac{7412}{9}\right], \tag{167}$$

$$c_{P4}^-=-32(n+2)\left[\frac{2\pi^4}{15}(5n+22)+\frac{2045}{216}n^2+\frac{824\,537}{1458}n+\frac{1\,775\,531}{729}+6(n^2+10n+52)\zeta(3)-\tau(n)\right]. \tag{168}$$

Equations (156)–(168) are the basis for future Borel resummations and variational calculations of the functions $P_\pm(1,u,3)$ for general n.

Once the resummed functions $F_\pm(1,u,3)$ and $P_\pm(1,u,3)$ are known, they can be used to calculate $f_\pm^{(0,0)}(1,u,3)$ and $f_\pm^{(1,0)}(1,u,3)$ via the integral representations, Eqs. (142) and (143), for general $u>0$. This is of relevance for an analysis in the nonasymptotic region.

VI. ASYMPTOTIC AMPLITUDES

In the following, we derive the perturbative four-loop expressions for the asymptotic quantities $f_s^\pm(t)$, $f_{ns}(t)$, $A^\pm$, ξ_+, and R_ξ^+ defined in the Introduction by applying the results of the preceding sections to the asymptotic critical region $\xi_\pm \gg \mu^{-1}$ corresponding to the limit $l_\pm\to 0$, $u(l_\pm)\to u(0)\equiv u^*$. Although the amplitude functions $f_\pm^{(0,0)}(1,u,d)$, $f_\pm^{(1,0)}(1,u,d)$, and $q_\pm(1,u,d)$ do not have an expansion around $u=0$ in integer powers of u, there exist important simplifications for the structure of these amplitude functions near the fixed point, where they have a convergent expansion in integer powers of $u-u^*$. In particular, at the fixed point $u=u^*$, where $\beta_u(u^*,\varepsilon)=0$ the differential equations (137)–(139) are reduced to the exact relations

$$q_+(1,u^*,d)=2\nu P_+(1,u^*,d), \tag{169}$$

$$f_{\pm}^{(0,0)}(1,u^*,d) = -\frac{P_{\pm}^*}{d}[f_{\pm}^{(1,0)}(1,u^*,d) + 2\nu^2 P_{\pm}^* B(u^*)], \tag{170}$$

$$f_{\pm}^{(1,0)}(1,u^*,d) = \frac{\nu P_{\pm}^*}{1-\alpha}[F_{\pm}(1,u^*,d) + 4\nu B(u^*)] \tag{171}$$

with $P_{\pm}^* \equiv P_{\pm}(1,u^*,d)$. Equations (169)–(171) can also be derived from Eqs. (141)–(143) after substituting $\beta_u(u',\varepsilon) = \omega(u'-u^*) + O([u'-u^*]^2)$. The critical exponents α and ν are determined by the fixed-point value of $\zeta_r(u)$,

$$\nu = [2 - \zeta_r(u^*)]^{-1}, \tag{172}$$

$$\alpha = 2 - d\nu = \frac{\varepsilon - 2\zeta_r(u^*)}{2-\zeta_r(u^*)}. \tag{173}$$

In addition, we have from Eqs. (131) and (132),

$$q_-(1,u^*,d) = 2\nu P_-(1,u^*,d), \tag{174}$$

$$P_-(1,u^*,d) = -\frac{3}{4}\nu^{-1} + P_+(1,u^*,d). \tag{175}$$

This implies that $f_{\pm}^{(0,0)}(1,u^*,d)$, $f_{\pm}^{(1,0)}(1,u^*,d)$, and $q_{\pm}(1,u^*,d)$ can be expressed by $P_{\pm}(1,u^*,d)$ and $F_{\pm}(1,u^*,d)$ which have an expansion in integer powers of u^* and are Borel resummable. As an important consequence, the various asymptotic amplitudes defined in the Introduction have perturbative expansions in integer powers of u^* that are Borel resummable.

We point out that, within a perturbative treatment, it would be an inappropriate procedure to derive the fixed-point values of q_+, $f_{\pm}^{(0,0)}$, and $f_{\pm}^{(1,0)}$ by first expanding the right-hand side of Eqs. (137)–(140) with respect to u at $u \neq u^*$ and then by substituting the (approximate) Borel-resummed fixed-point value u^*. The reason is that the terms proportional to $\beta_u(u,\varepsilon)$ in Eqs. (137)–(140) must vanish *exactly* at $u = u^*$ because $\beta_u(u^*,\varepsilon) = 0$. This property would be destroyed in a truncated perturbative treatment.

A. Correlation length

The relation between $\xi_{\pm}(t)$ and the reduced temperature t is determined implicitly by [30,31,34]

$$r = at = \xi_{\pm}^{-2} q_{\pm}(1,u(l_{\pm}),d)\exp\left(\int_{l_{\pm}}^{1}\zeta_r(u(l'))\frac{dl'}{l'}\right), \tag{176}$$

where $a = Z_r(u,\varepsilon)^{-1}a_0$. The asymptotic form of the correlation lengths follows from Eqs. (172) and (176) as

$$\xi_{\pm}(t) = \xi_0^{\pm}|t|^{-\nu}. \tag{177}$$

After the choice

$$\mu^{-1} = \xi_0^+, \tag{178}$$

the square of the nonuniversal amplitude ξ_0^+ is given (at infinite cutoff) as a function of u, u^*, and a_0 for $2<d<4$ by [30,31]

$$(\xi_0^+)^2 = Z_r(u,\varepsilon)a_0^{-1}2\nu P_+(1,u^*,d)\tilde{C} \tag{179}$$

with

$$\tilde{C} = \exp\left(\int_u^{u^*}\frac{\zeta_r(u^*) - \zeta_r(u')}{\beta_u(u',\varepsilon)}du'\right), \tag{180}$$

where we have used Eq. (169). The asymptotic amplitude of the pseudocorrelation length ξ_- below T_c reads [31]

$$\xi_0^- = \left(\frac{Q_-^*}{2Q_+^*}\right)^{\nu}\xi_0^+ = \left(\frac{-q_-^*}{q_+^*}\right)^{\nu}\xi_0^+ = \left(\frac{-P_-^*}{P_+^*}\right)^{\nu}\xi_0^+, \tag{181}$$

with $P_{\pm}^* \equiv P_{\pm}(1,u^*,d)$, $q_{\pm}^* \equiv q_{\pm}(1,u^*,d)$ where we have used Eqs. (123), (124), (131), (132), (169), and (174). We note that the ratio

$$\frac{\xi_0^+}{\xi_0^-} = \left(\frac{2\nu P_+^*}{(3/2) - 2\nu P_+^*}\right)^{\nu} \tag{182}$$

is not a universal quantity, since the pseudocorrelation length ξ_- is not a physical correlation length below T_c. In three dimensions, the power series for $P_{\pm}(1,u,3)$ are given in Eqs. (156)–(168) up to four-loop order.

B. Singular part of the free energy

Evaluating Eq. (133) asymptotically (see Appendix E) we find the singular part of the free energy for both $\alpha>0$ and $\alpha<0$,

$$f_s^{\pm}(t) = A_d[\xi_{\pm}(t)]^{-d}\left\{f_{\pm}^{(0,0)}(1,u^*,d) - \frac{\nu}{2\alpha}B(u^*) \times[2\nu P_{\pm}(1,u^*,d)]^2\right\}, \tag{183}$$

where we have used Eqs. (136), (169), and (174). Equations (183), (170), and (171) yield

$$f_s^{\pm}(t) = A_d[\xi_{\pm}(t)]^{-d}\left\{-\nu^2 P_{\pm}(1,u^*,d)^2 \times\frac{4\nu B(u^*) + \alpha F_{\pm}(1,u^*,d)}{\alpha(1-\alpha)(2-\alpha)}\right\}. \tag{184}$$

Defining

$$(R_{\xi}^{\pm})^d = -\alpha(1-\alpha)(2-\alpha)(\xi_{\pm})^d f_s^{\pm}, \tag{185}$$

we obtain

$$(R_{\xi}^{\pm})^d = A_d\nu^2 P_{\pm}(1,u^*,d)^2[4\nu B(u^*) + \alpha F_{\pm}(1,u^*,d)], \tag{186}$$

$$A^{\pm}=(R_{\xi}^{\pm}/\xi_0^{\pm})^d, \tag{187}$$

and

$$\frac{A^+}{A^-}=\left[\frac{P_+^*}{-P_-^*}\right]^{\alpha}\frac{4\nu B(u^*)+\alpha F_+(1,u^*,d)}{4\nu B(u^*)+\alpha F_-(1,u^*,d)}. \tag{188}$$

Only R_ξ^+ and A^+/A^- are universal. The quantity $R_\xi^- =(A^-/A^+)^{1/d}(\xi_0^-/\xi_0^+)R_\xi^+$ is nonuniversal since $\xi_0^+/\xi_0^- =\xi_+/\xi_-$ is nonuniversal [see Eq. (182)]. Equations (186) and (188) have expansions in integer powers of u^*, since the functions $F_\pm(1,u^*,d)$, $P_\pm(1,u^*,d)$, $B(u^*)$, $\nu =[2-\zeta_r(u^*)]^{-1}$, and $\alpha=2-d[2-\zeta_r(u^*)]^{-1}$ have such expansions. These expansions, however, are not convergent and need to be resummed. In three dimensions, the power series for $F_\pm(1,u,3)$ are given in Eqs. (145)–(155) up to four-loop order.

C. Regular part of the free energy

The regular part $f_{ns}(t)$, Eq. (3), contains contributions from both the background Hamiltonian, Eq. (15), and from fluctuation-induced regular terms of the Gibbs free energy F_0, Eq. (14). The latter terms depend on the cutoff. From dimensional arguments, one finds that f_0 and f_1 diverge as Λ^d and Λ^{d-2} for $\Lambda\to\infty$, whereas $B_{cr}(\Lambda)$ has a finite limit $B_{cr}(\infty)$ for $d<4$. Within the dimensional-regularization scheme, the cutoff dependence is ignored. After the subtraction of $\delta\Gamma_0$ in Eq. (29), there is only the cutoff-independent part $-\frac{1}{2}B_{cr}(\infty)t^2$ of the regular term $-\frac{1}{2}Bt^2$ that is still contained in $\mathring{\Gamma}(r_0',u_0,d)$, Eq. (29), and in $\mathring{\mathcal{F}}_\pm(\xi_\pm(t),u_0,d)$, Eq. (133). The corresponding analytic expression for the fluctuation-induced coefficient B_{cr} has been calculated previously within the minimal subtraction scheme at fixed $d<4$ in Ref. [31] for $\alpha<0$ and by Dohm and Schloms for $\alpha>0$ as quoted in Ref. [59]. Since no derivation was given in Refs. [31,59] and since different results $B^+\neq B^-$ were obtained in Ref. [13], we derive the coefficient B_{cr} in Appendix E. The results read, at $\Lambda=\infty$ [31,59]

$$B_{cr}^+=B_{cr}^-=-(\xi_0^+)^{\varepsilon}A_d a^2\int_0^{u^*}\frac{B(u')}{\beta_u(u',\varepsilon)}\left[\exp\left(\int_u^{u'}\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}du''\right)\right]du' \quad (\alpha<0), \tag{189}$$

$$B_{cr}^+=B_{cr}^-=-(\xi_0^+)^{\varepsilon}A_d a^2\frac{\nu}{\alpha}\int_0^{u^*}\frac{2B(u')[\zeta_r(u')-\zeta_r^*]+\beta_u(u',\varepsilon)\partial B(u')/\partial u'}{\beta_u(u',\varepsilon)}\left[\exp\left(\int_u^{u'}\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}du''\right)\right]du' \quad (\alpha>0), \tag{190}$$

with $a=Z_r(u,\varepsilon)^{-1}a_0$. The quantities $B_{cr}^\pm$ are defined by

$$B_{cr}^+=\lim_{t\to 0+}[C^+(t)-C_s^+(t)-C_B], \tag{191}$$

$$B_{cr}^-=\lim_{t\to 0-}[C^-(t)-C_s^-(t)-C_B], \tag{192}$$

where $C_s^\pm(t)$ is the singular part of the specific heat $C^\pm(t)$ and C_B is the background contribution from degrees of freedom other than $\boldsymbol{\varphi}_0(\mathbf{x})$. Equations (189) and (190) are valid for the φ^4 field theory in $2<d<4$ dimensions within the minimal subtraction scheme at $\Lambda=\infty$. The continuity of $C^\pm(t)$ at T_c for $\alpha<0$ follows from the fact that, according to Eq. (189), $B_{cr}^+=B_{cr}^-=B_{cr}$ is entirely determined by the field-theoretic functions $\beta_u(u,\varepsilon),\zeta_r(u)$, and $B(u)$ and by $Z_r(u,\varepsilon)$, which are identical above and below T_c within the minimal subtraction scheme.

Our results, however, do not yet exclude the possibility that $B_{cr}^+(\Lambda)\neq B_{cr}^-(\Lambda)$ within the φ^4 theory *at finite cutoff*. We do not know of a general thermodynamic argument in support of or against the continuity of the specific heat at T_c for $\alpha<0$ in real systems with short-range (and possibly subleading long-range) interactions.

One expects that the result $B_{cr}^+(\infty)=B_{cr}^-(\infty)$ should be obtained also within other renormalization schemes, provided that the limit $\Lambda\to\infty$ is taken. (This is consistent with the values for X_6 in Tables I and II of Ref. [40].) This expectation appears to be at variance with Fig. 3(b) of Ref. [13], where an apparent discontinuity $\lim_{t\to 0+}C^+(t) >\lim_{t\to 0-}C^-(t)$ of the specific heat at T_c is shown for $\alpha<0$ on the basis of Ref. [60], where the φ^4 field theory is treated at $\Lambda=\infty$, after additive renormalizations of the free energy. Unlike our renormalization scheme, however, the additive renormalizations of Ref. [60] are regular subtractions at finite cutoff, which are defined by the bare free energy at $t=1$ and $t=-1$, thus these subtractions are different above and below T_c. This asymmetric renormalization procedure introduces different t-independent contributions $-A^+/\alpha\neq -A^-/\alpha$ into the definition of the *renormalized* specific heat

$$C_b^R=\frac{A^\pm}{\alpha}|t|^{-\alpha}-\frac{A^\pm}{\alpha} \tag{193}$$

of Eq. (4.14) of Ref. [60].

The *total physical* specific heat $C^\pm(t)$, however, that is plotted in Fig. 3(b) of Ref. [13] should include *all* subtracted parts $\sim t^2$ of the total bare free energy above and below T_c. These parts that are cutoff dependent may well cancel the asymmetry of the t-independent parts of Eq. (193). Our result, Eq. (189), indeed suggests that the different constants

$-A^+/\alpha$ and $-A^-/\alpha$ of Eq. (193) are only a spurious effect of the renormalization procedure of Refs. [13,60]. No explicit analytic expressions were given for B^+ and B^- in Ref. [13] and no complete analysis of the problem of the possible discontinuity at T_c was performed. A conclusive answer to this problem would require a calculation of the specific heat, at least of the fluctuation-induced term $\frac{1}{2}B_{cr}(\Lambda)t^2$, within the φ^4 field theory at finite cutoff. Furthermore, the effect of subleading long-range interactions (that do not change the universal bulk critical behavior) should be investigated, since it is not clear *a priori* whether such interactions affect the nonuniversal height of the finite cusp of the specific heat for $\alpha<0$.

VII. VARIATIONAL CALCULATION

The perturbative results of the preceding sections need to be resummed. In a subsequent paper [47] we shall report on corresponding Borel resummations. Here we confine ourselves to a brief application of the variational approach.

A. Wegner expansion and strong-coupling limit

The expansion of the various amplitude functions, such as

$$P_\pm(u)\equiv P_\pm(1,u,d)=\sum_{m=0}^{\infty} c^{\pm}_{Pm}u^m, \tag{194}$$

is a weak-coupling expansion that is known to be divergent for any $u\neq 0$. We are primarily interested in the approach of the effective coupling $u(l_\pm)$ to the fixed point $u(l_\pm)\to u(0)=u^*$ corresponding to the critical limit $T\to T_c$, $r_0-r_{0c}\to 0$, $\xi_\pm\to\infty$, or $l_\pm^{-1}=\mu\xi_\pm\to\infty$ at fixed bare coupling u_0. Near the fixed point, the amplitude functions are regular with a convergent expansion in integer powers of $u-u^*$. This yields a fundamentally different expansion

$$P_\pm(u(l_\pm))=P_\pm(u^*)+P'_\pm(u^*)[u(l_\pm)-u^*]+O([u(l_\pm)-u^*]^2), \tag{195}$$

where $u(l_\pm)$ can be further expanded according to the Wegner expansion [6]

$$u(l_\pm)=u^*+a_u l_\pm^{\omega}+O(l_\pm^{2\omega}) \tag{196}$$

with the Wegner exponent [6]

$$\omega=\frac{\Delta}{\nu}=\frac{\partial}{\partial u}\beta(u,\varepsilon)_{u=u^*} \tag{197}$$

and the Wegner correction amplitude [31]

$$a_u=(u-u^*)\exp\left(\int_u^{u^*}\left[\frac{1}{u'-u^*}-\frac{\omega}{\beta_u(u',\varepsilon)}\right]du'\right). \tag{198}$$

Unlike the weak-coupling expansion (194) in powers of u, the expansion of Eqs. (195)–(198) constitutes a convergent expansion of $P_\pm(u(l_\pm))$ in integer powers of $l_\pm^{\omega}$. (Additional terms would of course exist in the φ^4 field theory at finite cutoff and in the φ^4 lattice theory at finite lattice spacing as well as in theories including $\varphi^5,\varphi^6,\ldots$ terms in the Hamiltonian.) The effective coupling $u(l_\pm)$ is a function of $u_0\xi_\pm^{\varepsilon}$ as determined by the implicit equation

$$u(l_\pm)=u_0\xi_\pm^{\varepsilon}A_d Z_u(u(l_\pm),\varepsilon)^{-1}Z_\varphi(u(l_\pm),\varepsilon)^2. \tag{199}$$

Thus the physical critical limit $\xi_\pm\to\infty$ at fixed u_0 can formally be considered as a strong-coupling limit $u_0\to\infty$ at fixed $\xi_\pm$. Correspondingly, the Wegner expansion of Eqs. (195)–(198), if expressed in terms of $u_0\xi_\pm^{\varepsilon}$, can be considered as a convergent strong-coupling expansion. This is seen explicitly by expressing Eq. (195) in terms of the dimensionless bare coupling

$$\overline{u}_B=u_0\xi_\pm^{\varepsilon}A_d. \tag{200}$$

Using Eqs. (196), (199), and $l_\pm=(\mu\xi_\pm)^{-1}$ we obtain Eq. (195) in the form of a strong-coupling expansion

$$P_\pm(u(l_\pm))=P_\pm(u^*)+c_0^{\pm}\overline{u}_B^{-\omega/\varepsilon}+O(\overline{u}_B^{-2\omega/\varepsilon}) \tag{201}$$

with the Wegner amplitudes

$$c_0^{\pm}(u,u^*,d)=P'_\pm(u^*)a_u[Z_u(u,\varepsilon)Z_\varphi(u,\varepsilon)^{-2}u]^{\omega/\varepsilon} \tag{202}$$

above and below T_c. On the other hand, Eqs. (199) and (200) can be inverted to express $u(l_\pm)$ in powers of $\overline{u}_B$ as

$$u(l_\pm)=\sum_{m=1}^{\infty}\tilde{a}_{um}\overline{u}_B^{m}. \tag{203}$$

The coefficients $\tilde{a}_{um}$ are the same above and below T_c. Using the expansion (194) with u replaced by $u(l_\pm)$ and substituting Eq. (203) yields $P_\pm(u(l_\pm))$ in the form of the weak-coupling expansion

$$P_\pm(u(l_\pm))=\sum_{m=0}^{\infty}f_{\pm m}^{(P)}\overline{u}_B^{m}. \tag{204}$$

The expansions of the type (201) and (204) are crucial ingredients of the order-dependent mapping and variational approach of previous work [46,48–55] in the context of the φ^4 theory. Corresponding divergent weak-coupling and convergent strong-coupling expansions have been known for a long time in the context of the anharmonic oscillator [48–55,61–63]. A basic problem is to derive (approximate) expressions for the strong-coupling coefficients of Eq. (201) in terms of the weak-coupling coefficients of Eq. (204).

An important ingredient in the systematic solution of this problem is the introduction of a variational parameter, such as a shifted reference frequency Ω in the case of the anharmonic oscillator [48–54,62] or a shifted scale parameter K in the case of the φ^4 theory [46,54,55] corresponding to a shifted Gaussian part of the φ^4 Hamiltonian. This permits one to set up a strong-coupling expansion of the type of Eq. (201) by reexpanding the weak-coupling expansion (204), where the coefficients of the strong-coupling expansion can be calculated from the coefficients of the weak-coupling expansion. For the case of the φ^4 theory, the coefficients of the strong-coupling expansion depend on K and on the Wegner exponent ω. The K dependence can be absorbed by expressing the strong-coupling coefficients in terms of the scaled bare coupling constant

$$\hat{u}_B = \frac{\bar{u}_B}{K^{2\varepsilon/\omega}}. \tag{205}$$

In an exact theory with a strong-coupling expansion up to infinite order, the expanded quantity must be independent of the dummy parameter K and the appearance of K in the expansion parameters must sum up to a vanishing net effect. In a truncated series up to a finite order L, however, there exists a spurious dependence on K or on $\hat{u}_B$. Specifically, the leading term $P_\pm(u^*)$ of the strong-coupling expansion (201) is expressed in terms of the weak-coupling coefficients $f^{(P)}_{\pm m}$ of Eq. (204) up to order L as [46,54,55]

$$P^{(L)}_\pm(u^*) = \text{opt}_{\hat{u}_B}\left[\sum_{m=0}^{L} f^{(P)}_{\pm m}\hat{u}_B^m \sum_{j=0}^{L-m}\binom{-m\varepsilon/\omega}{j}(-1)^j\right]. \tag{206}$$

An analogous formula was derived in the context of the anharmonic oscillator [50]. An optimal order-dependent value K_L is then determined by a variational procedure with respect to $\hat{u}_B$ to minimize the error of the truncated strong-coupling series. In the following, the corresponding (L-dependent) optimal value of $\hat{u}_B$ will be denoted by

$$\hat{u}_B^* = \frac{\bar{u}_B}{K_L^{2\varepsilon/\omega}}. \tag{207}$$

This approach has been used recently to calculate critical exponents [38,54]. Very recently, this approach has been applied also to calculate universal amplitude ratios [55] on the basis of three-loop results [36].

In the following we extend these calculations up to four-loop order. Up to $L=4$, the expression on the right-hand side (rhs) of Eq. (206) reads explicitly

$$\sum_{m=0}^{4} f^{(P)}_{\pm m}\hat{u}_B^m \sum_{j=0}^{L-m}\binom{-m\varepsilon/\omega}{j}(-1)^j = f^{(P)}_{\pm 0} + \frac{f^{(P)}_{\pm 1}}{6}\rho(\rho+1)(\rho+2)\hat{u}_B + \rho(2\rho-1)f^{(P)}_{\pm 2}\hat{u}_B^2 + (3\rho-2)f^{(P)}_{\pm 3}\hat{u}_B^3 + f^{(P)}_{\pm 4}\hat{u}_B^4 \tag{208}$$

with

$$\rho = 1+\varepsilon/\omega. \tag{209}$$

In order to avoid repetitions of formulas of the type (208) we shall use the abbreviation

$$\Phi_4(\{x_m\},\hat{u}_B) \equiv x_0 + \frac{x_1}{6}\rho(\rho+1)(\rho+2)\hat{u}_B + \rho(2\rho-1)x_2\hat{u}_B^2 + (3\rho-2)x_3\hat{u}_B^3 + x_4\hat{u}_B^4. \tag{210}$$

Equation (210) can be used for the various amplitude functions simply by replacing x_m with the corresponding weak-coupling coefficients.

B. Wegner exponent ω

In a first step, the Wegner exponent ω itself must be determined by the variational procedure [55] up to four-loop order. Since $u(l_\pm)$ approaches the finite fixed-point value u^* in the limit $\bar{u}_B \to \infty$ we have

$$\lim_{\bar{u}_B\to\infty} \frac{d}{d\ln \bar{u}_B}\ln\frac{u(l_\pm)}{\bar{u}_B} = -1. \tag{211}$$

From Eq. (199) and from the Z factors of Ref. [35] we obtain the weak-coupling expansions up to four-loop order in three dimensions

$$\frac{u(l_\pm)}{\bar{u}_B} = 1-4(n+8)\bar{u}_B + 8[2n^2+41n+170]\bar{u}_B^2 - 8\left[8n^3+299n^2+\frac{9178}{3}n+\frac{26\,000}{3}+32\zeta(3)(5n+22)\right]\bar{u}_B^3 + \left[2560\zeta(5)\times(2n^2+55n+186)+128\zeta(3)(203n^2+2500n+7260)+256n^4+\frac{44\,956}{3}n^3+273\,856n^2+1\,862\,112n + \frac{12\,081\,440}{3} - \frac{64\pi^4}{15}(n+8)(5n+22)\right]\bar{u}_B^4 + O(\bar{u}_B^5) \tag{212}$$

and

$$W_4(\bar{u}_B) \equiv \frac{d}{d\ln\bar{u}_B}\ln\frac{u(l_\pm)}{\bar{u}_B} = \sum_{m=0}^{4} f_m\bar{u}_B^m + O(\bar{u}_B^5), \tag{213}$$

with the coefficients

$$f_0 = 0, \tag{214}$$

$$f_1 = -4(n+8), \tag{215}$$

$$f_2=16(n^2+25n+106), \tag{216}$$

$$f_3=-8\left[32\zeta(3)(5n+22)+8n^3+299n^2+\frac{9178}{3}n+\frac{26\,000}{3}\right], \tag{217}$$

$$f_4=16\left[16n^4+\frac{4063}{3}n^3+\frac{91\,048}{3}n^2+\frac{681\,352}{3}n+\frac{1\,601\,680}{3}+640\zeta(5)(2n^2+55n+186)+32\zeta(3)(163n^2+2004n+5852)-\frac{16\pi^4}{15}(n+8)(5n+22)\right]. \tag{218}$$

Rewriting the function (213) in the form of a K- and ω-dependent strong-coupling expansion and using Eqs. (206), and (208) we obtain in the limit $\overline{u}_B\to\infty$,

$$W_4(\infty)=\mathrm{opt}_{\hat{u}_B}[\Phi_4(\{f_m\},\hat{u}_B)]. \tag{219}$$

For a given ρ, the optimal value $\hat{u}_B^{\star}(\rho)$ is determined by the condition

$$0=\left(\frac{\partial\Phi_4(\{f_m\},\hat{u}_B)}{\partial\hat{u}_B}\right)_{\hat{u}_B=\hat{u}_B^{\star}(\rho)} \tag{220}$$

for an extremum of the rhs of Eq. (219) with respect to $\hat{u}_B$, i.e.,

$$0=\frac{f_1}{6}\rho(\rho+1)(\rho+2)+2\rho(2\rho-1)f_2\hat{u}_B^{\star}+3(3\rho-2)f_3(\hat{u}_B^{\star})^2+4f_4(\hat{u}_B^{*})^3. \tag{221}$$

Substituting the solution $\hat{u}_B^{*}(\rho)$ of Eq. (220) into the rhs of Eq. (219) and requiring the condition (211) in the form $W_4(\infty)=-1$ yields

$$\Phi_4(\{f_m\},\hat{u}_B^{\star}(\rho))=-1. \tag{222}$$

Equations (220) and (222) determine ρ and $\omega=(\rho-1)^{-1}$ implicitly. They can be solved numerically for arbitrary n. For example, for $n=0,1,2,3$ we find

$$\rho=2.386\,79,\quad \omega=0.721\,090\quad (n=0), \tag{223}$$

$$\rho=2.378\,07,\quad \omega=0.725\,653\quad (n=1), \tag{224}$$

$$\rho=2.367\,72,\quad \omega=0.731\,144\quad (n=2), \tag{225}$$

$$\rho=2.356\,29,\quad \omega=0.737\,305\quad (n=3). \tag{226}$$

These values are close to those obtained in three-loop order [55]. In the following we consider ρ as a known parameter.

C. Universal amplitude ratios A^+/A^- and $P=\alpha^{-1}(A^+/A^- -1)$

We rewrite Eq. (188) for $d=3$ in the form [35]

$$\frac{A^+}{A^-}=\left[\frac{2\nu P_+^{\star}}{(3/2)-2\nu P_+^{*}}\right]^{\alpha}\left[1-\alpha\frac{F_-(u^{\star})-F_+(u^{\star})}{4\nu B(u^{\star})+\alpha F_-(u^{\star})}\right]. \tag{227}$$

Since α, ν, $P_+^{\star}$, and $B(u^{\star})$ are already known with high accuracy [36], we consider only a variational calculation of $F_{\pm}(u^*)$ on the basis of our four-loop results.

The weak-coupling expansion of $F_-(u)$ has been given in Eqs. (145) and (151)–(155). Using Eq. (212), we obtain the weak-coupling expansion in terms of $\overline{u}_B$:

$$u(l_-)F_-(u(l_-))=\sum_{m=0}^{4} f_m^{(F)}\overline{u}_B^m, \tag{228}$$

with the coefficients up to four-loop order,

$$f_0^{(F)}=\frac{1}{2}, \tag{229}$$

$$f_1^{(F)}=-4, \tag{230}$$

$$f_2^{(F)}=8(n+26), \tag{231}$$

$$f_3^{(F)}=-480(3n+22)+c_{F3}^-, \tag{232}$$

$$f_4^{(F)}=-128n^3+6112n^2+\frac{422\,720}{3}n+\frac{1\,730\,560}{3}+1024(5n+22)\zeta(3)-12(n+8)c_{F3}^-+c_{F4}^-. \tag{233}$$

Application of the variational strong-coupling expansion yields

$$u^*F_-(u^*)=\mathrm{opt}_{\hat{u}_B}[\Phi_4(\{f_m^{(F)}\},\hat{u}_B)]. \tag{234}$$

For $0\leqslant n\lesssim 53.5$ the function $\Phi_4(\{f_m^{(F)}\},\hat{u}_B)$ is convex with only one minimum in the range $-\infty<\hat{u}_B<\infty$, thus the optimal value $\hat{u}_B^{\star}$ in this range is uniquely determined by $\partial\Phi_4(\{f_m^{(F)}\},\hat{u}_B^{\star})/\partial\hat{u}_B^{\star}=0$, and the fixed-point value $u^{\star}F_-(u^{\star})$ is given by

$$u^{\star}F_-(u^*)=\Phi_4(\{f_m^{(F)}\},\hat{u}_B^{*}) \tag{235}$$

$$=0.372\,144\quad\text{for}\quad n=2. \tag{236}$$

Our four-loop variational result for $u^{\star}F_-(u^{\star})$ in the range $0\leqslant n\leqslant 10$ is compared in Fig. 2 with the previous three-loop variational result [55] as well as with very recent Borel-resummed values based on our four-loop results for $n=2$ [47] and with earlier Borel-resummed five-loop results for $n=1$ [35]. We see that the four-loop variational result agrees with the Borel-resummed values within their error bars and constitutes a significant improvement over the three-loop variational result. Since $F_-(u^\star)$ enters A^+/A^- in the form of $\alpha F_-(u^\star)$, however, this improvement has only a small effect on A^+/A^-.

The same procedure can be applied to the difference $u[F_-(u)-F_+(u)]$. It has the weak-coupling expansion up to four-loop order

$$u(l_\pm)[F_-(u(l_-))-F_+(u(l_+))]=\sum_{m=0}^{4} f_m^{(\Delta F)}\overline{u_B^m}, \tag{237}$$

where the coefficients $f_m^{(\Delta F)}$ are obtained from Eqs. (145)–(155) and (212) as

$$f_0^{(\Delta F)}=\frac{1}{2}, \tag{238}$$

$$f_1^{(\Delta F)}=n-4, \tag{239}$$

$$f_2^{(\Delta F)}=-2n^2-20n+208, \tag{240}$$

$$f_3^{(\Delta F)}=168n^2-336n-10\,560+c_{F3}^- -c_{F3}^+, \tag{241}$$

$$f_4^{(\Delta F)}=32n^4-504n^3-\frac{21\,680}{3}n^2+\frac{259\,648}{3}n+\frac{1\,730\,560}{3} -256\zeta(3)(5n^2-2n+88)+12(n+8)(c_{F3}^+-c_{F3}^-) +c_{F4}^- -c_{F4}^+. \tag{242}$$

For $0\leqslant n \lesssim 2.75$, the function $\Phi_4(\{f_m^{(\Delta F)}\},\hat{u}_B)$ is convex with only one minimum in the range $-\infty<\hat{u}_B<\infty$, thus $\hat{u}_B^*$ is uniquely determined by $\partial\Phi_4(\{f_m^{(\Delta F)}\},\hat{u}_B^\star)/\partial\hat{u}_B^\star=0$. For $2.75 \lesssim n \lesssim 23.66$, one of the two turning points has to be chosen for reasons of continuity, and for $23.66 \lesssim n<\infty$, the function $\Phi_4(\{f_m^{(\Delta F)}\},\hat{u}_B)$ has a global maximum that determines $\hat{u}_B^\star$. In all cases, the fixed-point value $u^\star[F_-(u^\star)-F_+(u^\star)]$ is given by

$$u^\star[F_-(u^\star)-F_+(u^*)]=\Phi_4(\{f_m^{(\Delta F)}\},\hat{u}_B^\star) \tag{243}$$

$$=0.460\,535 \quad \text{for} \quad n=2. \tag{244}$$

A comparison analogous to that in Fig. 2 is shown in Fig. 3. There is only a small difference between the four-loop and three-loop variational results for $u^\star[F_-(u^\star)-F_+(u^\star)]$. According to Eq. (227) this implies only a small difference between the corresponding value for A^+/A^-.

The structure of Eq. (227) shows that there exists a close correlation between the value of α and of $1-A^+/A^-$. It has therefore been proposed [64] to study the universal combination

$$P=\alpha^{-1}(1-A^+/A^-), \tag{245}$$

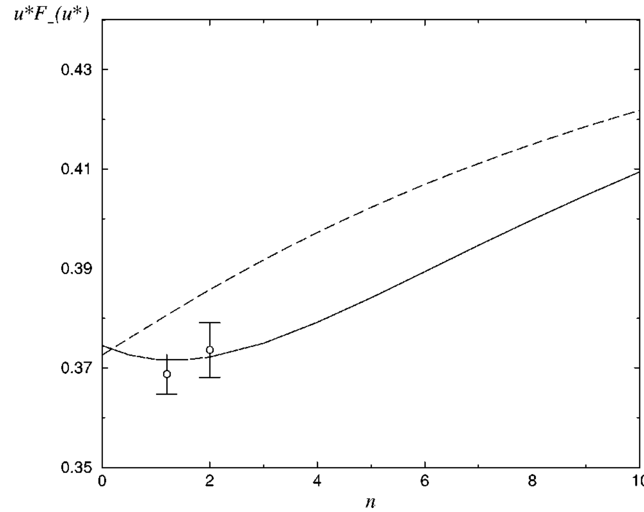

FIG. 2. Four-loop variational result (solid line) for the amplitude function $u^\star F_-^\star$ as a function of n. The dashed line represents the three-loop variational result of Ref. [55]. Also shown are the four-loop Borel-resummation result for $n=2$ (open circle, Ref. [47]) and the five-loop Borel-resummation result for $n=1$ (open circle, Ref. [35]).

which is not sensitive to the precise value of α. This is particularly important for the case $n=2$ because of the unresolved discrepancy between the numerical [11] and experimental [9,10] values of α.

Our variational calculation of P starts from Eq. (227) with [9,10] $\alpha=-0.010\,56$ and $\nu=(2-\alpha)/3$. For $B(u^\star)$, $P_+(u^\star)$, and $u^\star$ we use the Borel-resummed values of Ref. [35]. $F_-(u^\star)$ and $F_-(u^\star)-F_+(u^\star)$ are taken from Eqs. (236) and (244). The result according to Eq. (245) is

$$P_{\text{var}}=4.344. \tag{246}$$

This is to be compared with the numerical value for the XY model [11]

$$P_{\text{XY}}=4.3\pm0.2, \tag{247}$$

and with the experimental value for ^{4}He [10,65],

$$P_{\text{exp}}=4.19\pm0.25. \tag{248}$$

Our variational result Eq. (246) is in agreement with the numerical estimate Eq. (247) and with the experimental value Eq. (248) within the error bars. An estimate of error bars for the variational result is planned for future research.

Finally we note that new Borel resummations of P have been performed in a separate paper [47] on the basis of the four-loop series of the present paper. For $n=2$, the Borel-resummed value [47]

$$P_{\text{Borel}}=4.433\pm0.077 \tag{249}$$

has smaller error bars than those of P_{XY} and P_{exp} and than that of the previous Borel-resummed value based on the three-loop series [37]. Our variational result, Eq. (246), is slightly outside the error bar of the Borel-resummed value, Eq. (249).

D. Universal ratio R_ξ^+

For the purpose of calculating R_ξ^+, Eq. (186), in three dimensions, we first define the function

$$R_+(u)=(4\pi)^{-1}\nu(u)^2P_+(u)^2[4\nu(u)B(u)+\alpha(u)F_+(u)], \tag{250}$$

where $P_+(u)\equiv P_+(1,u,3)$ and

$$\nu(u)=[2-\zeta_r(u)]^{-1}, \tag{251}$$

$$\alpha(u)=2-3\nu(u). \tag{252}$$

Using the four-loop expansions of Eqs. (156)–(161), (145)–(150), and of $\zeta_r(u)$ and $B(u)$ [35] we get

$$\frac{4\pi R_+(u)}{n}=\frac{1}{8}+(n+2)u+(n+2)\left(\ln\frac{9}{16}-\frac{371}{54}+\frac{7}{2}n\right)u^2 +(n+2)\left[10n^2+\frac{41}{27}n+\frac{7549}{27}+\frac{244n+1736}{9}\times\ln\frac{3}{4}+2(n+8)\left\{6\zeta(3)+\pi^2+16\,\mathrm{Li}_2\times\left(-\frac{1}{3}\right)\right\}\right]u^3+O(u^4). \tag{253}$$

Together with Eq. (212), this yields the weak-coupling expansion in powers of $\overline{u}_B$,

$$4\pi R_+(u(l_+))=\sum_{m=0}^{3} f_{+m}^{(R)}\overline{u}_B^m+O(\overline{u}_B^4) \tag{254}$$

$$=\frac{n}{8}+n(n+2)\overline{u}_B-n(n+2)\left(\frac{n}{2}+\frac{2099}{54}+2\ln\frac{4}{3}\right)\overline{u}_B^2+n(n+2)\left[-2n^2+\frac{4333}{27}n+\frac{56\,141}{27}+\frac{100n+584}{9}\ln\frac{3}{4}+2(n+8)\times\left\{\pi^2+6\zeta(3)+16\,\mathrm{Li}_2\left(-\frac{1}{3}\right)\right\}\right]\overline{u}_B^3+O(\overline{u}_B^4). \tag{255}$$

For $u(l_+)\to u^*$ we obtain $R_+(u^\star)\equiv(R_\xi^+)^3$. Application of the variational strong-coupling expansion yields

$$(R_\xi^+)^3=(4\pi)^{-1}\mathrm{opt}_{\hat{u}_B}[\Phi_4(\{f_{+m}^{(R)}\},\hat{u}_B)]. \tag{256}$$

For $0\leqslant n\leqslant 66$ the function $\Phi_4(\{f_{+m}^{\{R}\},\hat{u}_B)$ has one turning point that determines $\hat{u}_B^\star$. The numerical values of our variational results for $n=1,2,3$ are

$$R_\xi^+=0.273,\quad n=1, \tag{257}$$

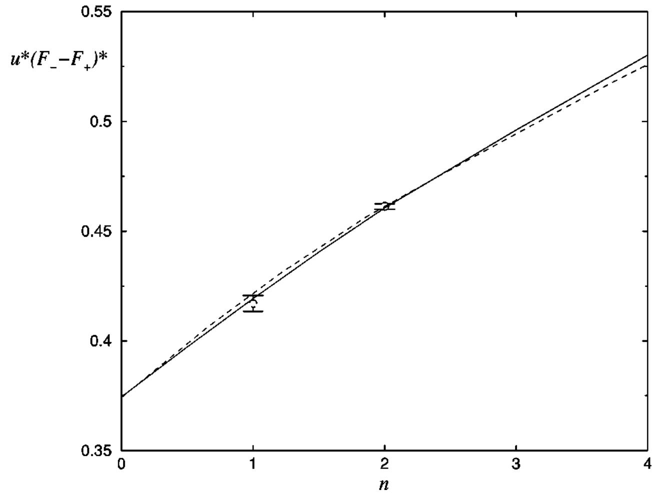

FIG. 3. Four-loop variational result (solid line) for the difference of the amplitude functions $u^\star(F_-^\star-F_+^\star)$ as a function of n. The dashed line represents the three-loop variational result of Ref. [55]. Also shown are the four-loop Borel-resummation result for $n=2$ (open circle, Ref. [47]) and the five-loop Borel-resummation result for $n=1$ (open circle, Ref. [35]).

$$R_\xi^+=0.366,\quad n=2, \tag{258}$$

$$R_\xi^+=0.441,\quad n=3. \tag{259}$$

They are close to earlier Borel-resummed values in Table IV of Ref. [22].

For $n\geqslant 67$, the function $\Phi_4(\{f_{+m}^{(R)}\},\hat{u}_B)$ has a maximum, a turning point, and a minimum. For reasons of continuity, the optimal value $\hat{u}_B^\star$ is uniquely determined by the maximum, since the resulting value of R_ξ^+ correctly approaches the exactly known limit $n\to\infty$, where [5,42]

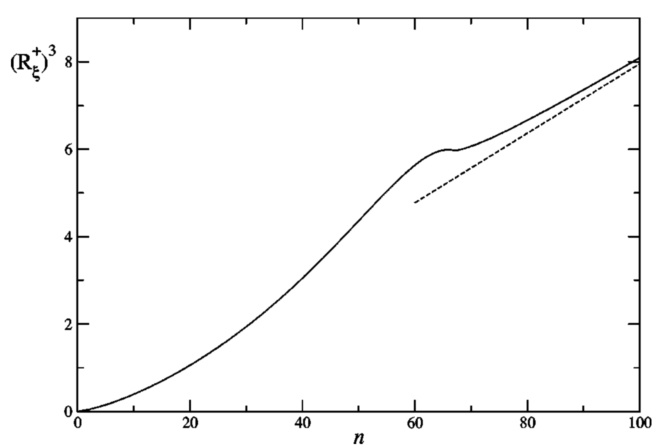

FIG. 4. Four-loop variational result (solid line) for the universal quantity $(R_\xi^+)^3$ defined by Eq. (9) as a function of n. For large n, the solid line approaches the dashed line $(4\pi)^{-1}n$ representing the exact large-n limit of $(R_\xi^+)^3$.

$$\lim_{n\to\infty}\frac{(R_\xi^+)^d}{n}=\frac{2^{d-2}\pi^{d/2}\Gamma(d/2)}{(d-2)^2\sin[\pi(4-d)/2]}=A_d \tag{260}$$

$$=\frac{1}{4\pi}\quad\text{for}\quad d=3 \tag{261}$$

with A_d given by Eq. (113). In Fig. 4 the four-loop variational result for $(R_\xi^+)^3$ is plotted for $0\leqslant n\leqslant 100$.

ACKNOWLEDGMENT

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APPENDIX A: ANALYTIC DEFINITION OF CONSTANTS

Here we present the analytic definition and numerical values of the following constants appearing in our four-loop results:

$$c_1=\int_0^1\frac{dx}{\sqrt{6-2x^2}}\left[\ln\frac{3}{4}+\ln\frac{3+x}{2+x}+\frac{x}{2+x}\left(\ln\frac{3+x}{3}+\frac{x}{2-x}\ln\frac{2+x}{4}\right)\right]=0.021\,737\,576\,3, \tag{A1}$$

$$c_2=\frac{\pi^2}{4\sqrt{2}}+\sqrt{2}\int_0^1\frac{dx}{\sqrt{1+x^2}}\left[\ln\frac{x}{1+x}+\frac{\ln(1+x)}{x}\right]$$
$$=2\sqrt{2}\left[\frac{\pi^2}{12}+\mathrm{Li}_2(1-\sqrt{2})+\mathrm{Li}_2(2\sqrt{2}-2)+\mathrm{Li}_2(-\sqrt{2})+\ln(\sqrt{2}-1)\ln(4-2\sqrt{2})\right]=0.973\,771\,427, \tag{A2}$$

$$c_3=\int_{\pi/4}^{\pi/2}\frac{d\theta}{\cos\theta-\sin\theta}\left[\ln\left(\tan\frac{\theta}{2}\right)\right]\ln(\tan\theta)=0.516\,197\,144, \tag{A3}$$

$$c_4=\frac{1}{2\pi}\int_0^\infty\frac{(\arctan x)^3}{1/4+x^2}\,\frac{dx}{x}=0.129\,107\,460, \tag{A4}$$

$$J_{m,n}^{(k)}=\frac{2^{k-1}}{\pi^{k+1}}\int_0^\infty\ln\frac{1+(p+r)^2}{1+(p-r)^2}\arctan\frac{p}{2}\left(\arctan\frac{r}{2}\right)^k\frac{dp\,dr}{(1+p^2)^m(1+r^2)^n}. \tag{A5}$$

The numerical values of $J_{m,n}^{(k)}$ are $J_{1,1}^{(0)}=0.219\,999\,124$, $J_{1,2}^{(0)}=0.068\,113\,807\,4$, $J_{2,1}^{(0)}=0.048\,385\,460\,5$, $J_{2,2}^{(0)}=0.019\,466\,092\,0$, $J_{1,1}^{(1)}=0.107\,718\,159$, $J_{2,1}^{(1)}=0.019\,475\,195\,7$, $J_{3,1}^{(1)}=0.007\,985\,463\,86$, $J_{2,2}^{(1)}=0.005\,045\,077\,68$. We define the following functions:

$$f_1(p,r,\varphi)=\sqrt{p^2+r^2+2pr\cos\varphi}, \tag{A6}$$

$$f_2(p,r,\varphi)=\sqrt{p^2+r^2+2pr\cos\varphi+4\sin^2\varphi}, \tag{A7}$$

$$f_3(p,r,\varphi)=\frac{prf_2(p,r,\varphi)}{8+2p^2+2r^2+2pr\cos\varphi}, \tag{A8}$$

$$f_4(p,r,\varphi)=\sqrt{1-2pr\cos\varphi+p^2r^2}, \tag{A9}$$

$$f_5(p,r,\varphi)=\sqrt{(1+p^2)r^2+(1+r^2)p^2+2pr(1+p^2)(1+r^2)\cos\varphi}, \tag{A10}$$

$$f_6(p,r,\varphi)=2+r^2+p^2. \tag{A11}$$

In the following we drop the arguments of these functions and use the abbreviation

$$\int\equiv\int_0^\infty dp\int_0^\infty dr\int_0^\pi d\varphi\sin\varphi \tag{A12}$$

for the integrals

$$A_1=\frac{2}{\pi^2}\int\frac{(\arctan f_3)^2}{f_2^2(1+p^2)(1+r^2)}=0.004\,969\,806\,804, \tag{A13}$$

$$A_2=\frac{2}{\pi^2}\int f_5^{-2}\left(\arctan\frac{f_5}{f_6}\right)^2=0.126\,705\,038\,5, \tag{A14}$$

$$A_3=\frac{2}{\pi^2}\int\left(\arctan\frac{\sqrt{1+2pr(1+r^2)\cos\varphi+r^2(2+p^2+r^2)}}{p}\right)^2\frac{(1+p^2)^{-1}}{1+2pr(1+r^2)\cos\varphi+r^2(2+p^2+r^2)}=0.503\,109\,256, \tag{A15}$$

$$A_4=\frac{2}{\pi^2}\int\frac{r^2}{f_1f_4f_5(1+f_1^2)}\left(\arctan\frac{f_4}{f_1}\right)\arctan\frac{f_5}{f_6}=0.107\,029\,9, \tag{A16}$$

$$A_5=\frac{1}{\pi}\int\frac{\arctan f_3}{f_2f_1(1+p^2)(1+r^2)}=0.043\,032\,747\,6, \tag{A17}$$

$$D_1=\frac{2}{\pi^2}\int\frac{r\,(\arctan f_3)\arctan(p/2)}{f_2(1+f_1^2)(1+p^2)^2(1+r^2)}=0.002\,366\,233\,043, \tag{A18}$$

$$D_2=\frac{1}{\pi}\int\frac{r\arctan f_3}{f_2(1+f_1^2)(1+p^2)^2(1+r^2)}=0.009\,399\,728\,17, \tag{A19}$$

$$D_3=\frac{2}{\pi^2}\int\frac{\arctan(f_1/2)}{f_1f_5(1+f_1^2)^2}\arctan\frac{f_5}{f_6}=0.052\,030\,993\,8, \tag{A20}$$

$$D_4=-\frac{2}{\pi^2}\int\frac{(2+f_1^2)}{f_4(1+f_1^2)^2}\left(\arctan\frac{f_4}{f_1}\right)\arctan\frac{f_1}{2}=-0.496\,521\,821, \tag{A21}$$

$$D_5=\frac{2}{\pi^2}\int\frac{1}{f_5(1+f_1^2)}\left(\frac{\arctan p}{p^3}-\frac{1}{p^2}\right)\arctan\frac{f_5}{f_6}=-0.051\,652\,015\,8, \tag{A22}$$

$$D_6=\frac{2}{\pi^2}\int\frac{1}{f_1f_4(1+f_1^2)}\left(\frac{\arctan p}{p^3}-\frac{1}{p^2}\right)\arctan\frac{f_4}{f_1}=-0.272\,886\,3, \tag{A23}$$

$$E_1=\frac{2}{\pi^2}\int\frac{(\arctan f_3)^2}{f_2^2(1+p^2)}=0.077\,537\,256\,112, \tag{A24}$$

$$E_1'=\frac{2}{\pi^2}\int\frac{(\arctan f_3)^2}{f_2^2(1+p^2)^2}=0.014\,449\,573\,28, \tag{A25}$$

$$E_1''=\frac{2}{\pi^2}\int\frac{(\arctan f_3)^2}{f_2^2(1+p^2)^3}=0.005\,958\,062\,001, \tag{A26}$$

$$E_2=\frac{1}{\pi}\int\frac{r\arctan f_3}{f_2(1+f_1^2)(1+r^2)}=0.124\,104\,75, \tag{A27}$$

$$E_3=\frac{2}{\pi^2}\int\frac{1}{f_1f_5}\arctan\frac{f_1}{2}\arctan\frac{f_5}{f_6}=0.350\,599\,930, \tag{A28}$$

$$E_4=\frac{2}{\pi^2}\int\frac{r^2}{f_1f_4f_5}\left(\arctan\frac{f_4}{f_1}\right)\arctan\frac{f_5}{f_6}=0.491\,503\,45, \tag{A29}$$

$$E_5=\frac{1}{\pi}\int \frac{\arctan f_3}{f_1 f_2(1+p^2)}=0.249\,720\,955, \tag{A30}$$

$$E_6=\frac{2}{\pi^2}\int\left(\arctan\frac{\sqrt{1+2pr(1+r^2)\cos\varphi+r^2(2+p^2+r^2)}}{p}\right)^2\frac{r^2\,(1+p^2)^{-1}}{1+2pr(1+r^2)\cos\varphi+r^2(2+p^2+r^2)}=0.853\,709\,186, \tag{A31}$$

$$F_1=\frac{2}{\pi^2}\int \frac{r\,(\arctan f_3)\,\arctan(p/2)}{f_2(1+f_1^2)(1+p^2)(1+r^2)}=0.007\,861\,658\,73, \tag{A32}$$

$$F_2=\frac{1}{\pi}\int \frac{r\arctan f_3}{f_2(1+f_1^2)(1+p^2)(1+r^2)}=0.021\,380\,900\,0, \tag{A33}$$

$$F_3=\frac{2}{\pi^2}\int \frac{\arctan(f_1/2)}{f_1 f_5(1+f_1^2)}\arctan\frac{f_5}{f_6}=0.096\,345\,809, \tag{A34}$$

$$F_4=\frac{2}{\pi^2}\int \frac{\arctan p}{f_5(1+f_1^2)\,p}\arctan\frac{f_5}{f_6}=0.177\,271\,421\,7, \tag{A35}$$

$$F_5=\frac{2}{\pi^2}\int \frac{\arctan p}{f_1 f_4(1+f_1^2)\,p}\arctan\frac{f_4}{f_1}=0.874\,819\,4, \tag{A36}$$

$$F_6=-\frac{2}{\pi^2}\int \frac{1}{f_4(1+f_1^2)}\left(\arctan\frac{f_4}{f_1}\right)\arctan\frac{f_1}{2}=-0.390\,967\,182, \tag{A37}$$

$$G_1=\frac{(4\pi)^4}{(2\pi)^{12}}\int d^3p\int d^3q\int d^3r\int d^3s\frac{1}{(1+p^2)(1+q^2)(1+r^2)(1+s^2)}$$
$$\times\frac{1}{(1+|\mathbf{p}+\mathbf{q}|^2)(1+|\mathbf{q}+\mathbf{r}|^2)(1+|\mathbf{r}+\mathbf{s}|^2)(1+|\mathbf{s}+\mathbf{p}|^2)(1+|\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s}|^2)}=0.000\,765\,3, \tag{A38}$$

$$G_2=\frac{(4\pi)^4}{(2\pi)^{12}}\int d^3p\int d^3q\int d^3r\int d^3s\frac{p^{-2}\,q^{-2}\,r^{-2}\,s^{-2}\,(1+|\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{s}|^2)^{-1}}{(1+|\mathbf{p}+\mathbf{q}|^2)(1+|\mathbf{q}+\mathbf{r}|^2)(1+|\mathbf{r}+\mathbf{s}|^2)(1+|\mathbf{s}+\mathbf{p}|^2)}=0.045\,54, \tag{A39}$$

$$G_3=\frac{(4\pi)^4}{(2\pi)^{12}}\int d^3p\int d^3q\int d^3r\int d^3s\frac{p^{-2}q^{-2}|\mathbf{p}+\mathbf{r}+\mathbf{s}|^{-2}|\mathbf{q}+\mathbf{r}+\mathbf{s}|^{-2}}{r^2\,s^2\,(1+|\mathbf{p}-\mathbf{q}|^2)(1+|\mathbf{q}+\mathbf{r}|^2)}\left(\frac{1}{1+|\mathbf{q}+\mathbf{s}|^2}-1\right)=-0.1695, \tag{A40}$$

$$H_1=\frac{2}{\pi^2}\int \frac{(\arctan f_3)^2}{f_2^2(1+f_1^2)(1+p^2)(1+r^2)}=0.000\,913\,765\,362, \tag{A41}$$

$$H_2=\frac{2}{\pi^2}\int \frac{1}{f_5^2(1+f_1^2)}\left(\arctan\frac{f_5}{f_6}\right)^2=0.058\,898\,731\,410, \tag{A42}$$

$$H_3=\frac{1}{\pi}\int \frac{\arctan f_3}{f_1 f_2(1+f_1^2)(1+p^2)(1+r^2)}=0.017\,101\,187\,5, \tag{A43}$$

$$H_4=\frac{2}{\pi^2}\int \frac{f_1}{f_4 f_5(1+f_1^2)}\left(\arctan\frac{f_4}{f_1}\right)\arctan\frac{f_5}{f_6}=-0.138\,602\,907\,09, \tag{A44}$$

$$H_5 = -\frac{2}{\pi^2}\int \frac{1}{f_4^2(1+f_1^2)}\left(\arctan\frac{f_4}{f_1}\right)^2 = -0.418\,219\,262\,0. \tag{A45}$$

Several of the integrals (A13)–(A45) are combined to define the quantities

$$X_{410} = 72E_2 + 48E_3 + 32E_4 + 96E_5 + 8E_6, \tag{A46}$$

$$X_{420} = 24E_6 + 12A_2 + 16A_3 + 48A_4 + 144A_5 + 216F_2 + 144F_3 + 96F_4 + 32F_5 + 24F_6, \tag{A47}$$

$$X_{430} = 24A_3 + 162D_2 + 108D_3 + 18D_4 + 24D_5 + 8D_6 + 18E_6 + 18G_2 + \tfrac{4}{3}G_3 + 36H_2 + 72H_3 + 24H_4 + 4H_5, \tag{A48}$$

which are used in Eqs. (43)–(52).

Finally we give the definition of several functions whose numerical values are listed in the standard literature. The function polylog(n,x) is defined as

$$\text{polylog}(n,x) = \sum_{k=0}^{\infty} \frac{x^k}{k^n}. \tag{A49}$$

It is also given recursively by

$$\text{polylog}(n,x) = \int_0^x \frac{\text{polylog}(n-1,t)}{t}\,dt \tag{A50}$$

with

$$\text{polylog}(1,x) \equiv -\ln(1-x). \tag{A51}$$

Equations (A49)–(A51) also yield definitions for the dilogarithmic function

$$\text{Li}_2(x) \equiv \text{polylog}(2,x) = \int_x^0 \frac{\ln(1-t)}{t}\,dt \tag{A52}$$

and the Riemann ζ function

$$\zeta(n) \equiv \text{polylog}(n,1) = \sum_{k=0}^{\infty} \frac{1}{k^n}. \tag{A53}$$

For $x>0$, $\text{Li}_2(x)$ satisfies the relations

$$\text{Li}_2(1-x) + \text{Li}_2\left(1-\frac{1}{x}\right) + \tfrac{1}{2}(\ln x)^2 = 0, \tag{A54}$$

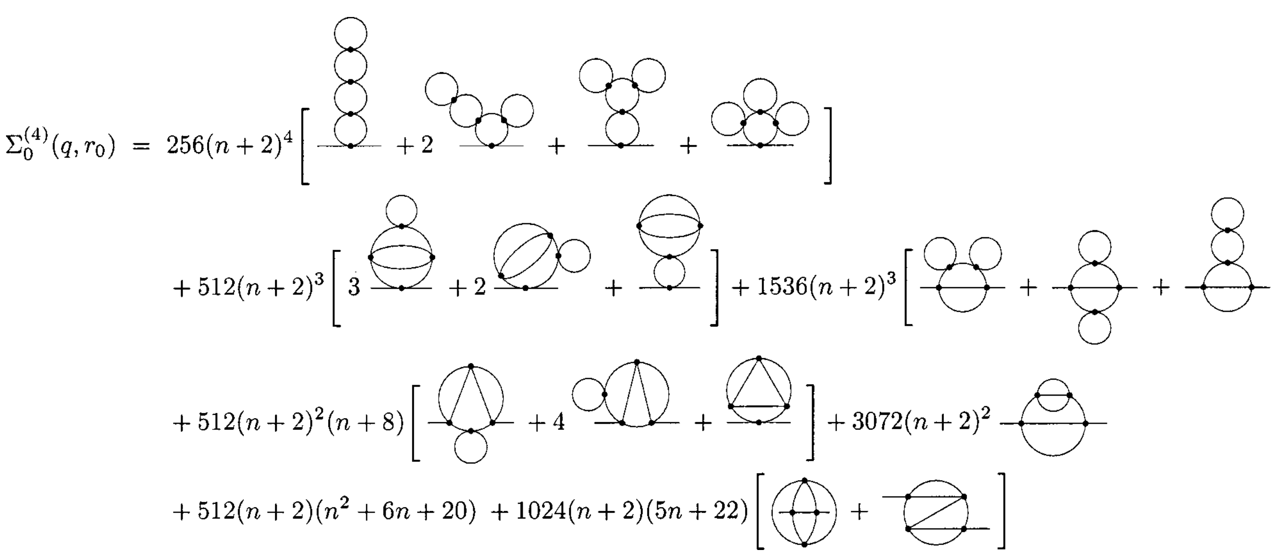
$$\text{Li}_2(-x) + \text{Li}_2\left(-\frac{1}{x}\right) + \frac{1}{2}(\ln x)^2 + \frac{\pi^2}{6} = 0. \tag{A55}$$

APPENDIX B: CORRELATION LENGTH IN FOUR-LOOP ORDER

In this appendix we sketch the derivation of Eqs. (61) and (68) and present the analytic four-loop expression for the amplitude function $Q_+(1,u,3)$. The self-energy above T_c,

$$\Sigma_0(q,r_0,u_0) = \sum_{m=1}^{\infty} (-u_0)^m \Sigma_0^{(m)}(q,r_0), \tag{B1}$$

in Eq. (54) is the sum of all one-particle irreducible m-loop diagrams with two (amputated) external legs. In Appendix B of Ref. [36] the diagrammatic contributions to Σ_0 are given in the three-loop approximation. The diagrammatic four-loop contributions are given by

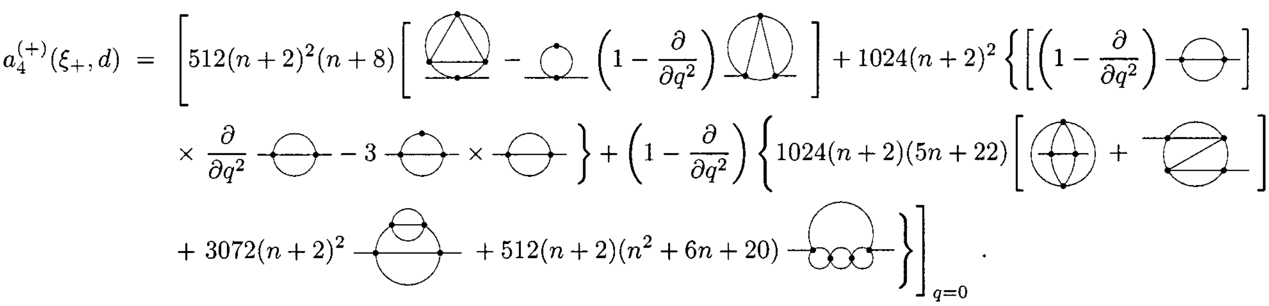


(B2)

For $q=0$, the lines denote the standard propagator $(r_0+p^2)^{-1}$ above T_c. From Eq. (B2) we get $\partial\Sigma_0^{(4)}/\partial q^2|_{q=0}$ in order to derive $\xi_+(r_0,u_0,d)$ up to four-loop order. We invert $\xi_+(r_0,u_0,d)$, which yields

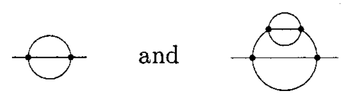
$$r_0(\xi_+,u_0,d)=\xi_+^{-2}\sum_{m=0}^{\infty} a_m^{(+)}(\xi_+,d)u_0^m. \tag{B3}$$

The four-loop coefficient of Eq. (B3) can be obtained from Eq. (B2),



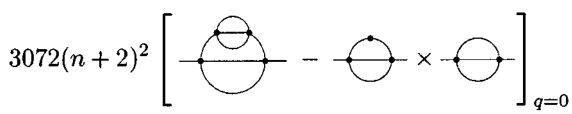
(B4)

The masses in the propagators of Eq. (B4) are ξ_+^{-2} rather than r_0. In evaluating Eq. (B4) for $d\to 3$, the contributions

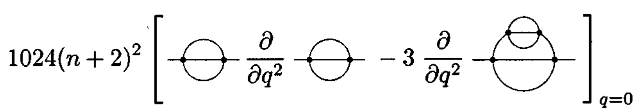


(B5)

are somewhat problematic as they are plagued by pole terms $\sim(d-3)^{-1}$. These poles, which cancel each other at the end of the calculation, yield finite contributions for $d\to 3$ that can easily be overlooked. Therefore we explicitly present the $d=3$ result of the following combinations of diagrams:



$$= \frac{(n+2)^2}{2\pi^4}\,\xi_+^{-2}\left\{4-3\pi^2+8\ln\frac{4}{3}+40\ln\frac{5}{4} - 9\left(\ln\frac{3}{4}\right)^2 - 18\left[\mathrm{Li}_2\left(-\tfrac{1}{4}\right)+\mathrm{Li}_2\left(-\tfrac{2}{3}\right)\right]\right\}, \tag{B6}$$



$$= \frac{2(n+2)^2}{27\pi^4}\,\xi_+^{-4}\left(37\ln\frac{5}{4}+\ln\frac{3}{4}-\frac{17}{2}\right). \tag{B7}$$

Together with the previous two-loop [34] and three-loop [36] calculations, this leads to the four-loop results for the functions $r_0'(\xi_+,u_0)>0$, $r_0'(\xi_-,u_0)<0$, and $\mathring{h}(\xi_+,u_0,3)$, with the new coefficients Eqs. (61) and (68).

From the four-loop expression of $\mathring{h}(\xi_+,u_0,3)$, Eq. (65), we obtain the amplitude function $Q_+(\mu\xi_+,u,3)$ according to Eq. (117). At $\mu\xi_+=1$ it has the expansion

$$Q_+(1,u,3)=1+\sum_{m=0}^{2} c_{Qm}u^{m+2}+32(n+2)\ln(96\pi u)\sum_{m=0}^{2}\tilde{c}_{Qm}u^{m+2}+O(u^5,u^5\ln u) \tag{B8}$$

with

$$\tilde{c}_{Q0}=1, \tag{B9}$$

$$\tilde{c}_{Q1}=4(n+14), \tag{B10}$$

$$\tilde{c}_{Q2}=4(4n^2+69n+458), \tag{B11}$$

$$c_{Q0}=(n+2)\left[68+\frac{16}{27}\right]+16\pi^2[C(n)-\tilde{S}_2(1,n)], \tag{B12}$$

$$c_{Q1}=\frac{16}{27}(n+2)\left[247n+3905-288(n+8)\mathrm{Li}_2\left(-\frac{1}{4}\right)-24(n+8)\pi^2\right]+4(43n+182)\ln\frac{4}{3}+64\pi^2(n+14)[C(n)-\tilde{S}_2(1,n)], \tag{B13}$$

$$c_{Q2}=32(n+2)\left\{\frac{32}{3}(5n+22)\left(4J_{1,1}^{(1)}-2J_{2,1}^{(1)}+4J_{3,1}^{(1)}+3E_1-E_1'+4E_1''+\frac{\pi^4}{160}+\frac{1}{12}\left[\mathrm{Li}_2\left(\frac{1}{3}\right)-\mathrm{Li}_2\left(-\frac{1}{3}\right)-\frac{(\ln 2)^2}{2}\right]\right)\right.$$
$$-\frac{8}{27}(173n+178)\left[\frac{1}{2}\left(\ln\frac{3}{4}\right)^2+\mathrm{Li}_2\left(-\frac{1}{3}\right)+\mathrm{Li}_2\left(\frac{1}{3}\right)\right]+32(n^2+6n+20)c_4-\frac{\pi^2}{27}(137n^2+2384n+9548)+\frac{2}{9}$$
$$\times(797n+1542)\ln\frac{5}{3}-\frac{32}{27}(60n^2+869n+3634)\mathrm{Li}_2\left(-\frac{1}{3}\right)+64\pi^2(4n^2+69n+458)[C(n)-\tilde{S}_2(1,n)]$$
$$\left.+3(n^2+50n+244)\zeta(3)+\frac{1}{27}(944n^2+12\,030n+33\,796)\ln\frac{4}{3}+\frac{4001}{144}n^2+\frac{3\,489\,713}{5832}n+\frac{9\,505\,097}{2916}\right\}. \tag{B14}$$

The logarithmic terms in Eq. (B8) arise from $r_{0c}(u_0,\varepsilon)$, Eq. (20), for $l=2$ in the limit $\varepsilon\to 1$.

APPENDIX C: BARE SPECIFIC HEAT ABOVE AND BELOW T_c

The singular part $C_s^{\pm}(t)$ and the cutoff-independent term B_{cr} of the specific heat in three dimensions are determined by the bare Gibbs free energy $\mathring{\mathcal{F}}_{\pm}(r_0',u_0)$ according to

$$C_s^{\pm}(t)+B_{cr}=-a_0^2\mathring{\Gamma}_{\pm}^{(2,0)}(r_0',u_0), \tag{C1}$$

$$\mathring{\Gamma}_{\pm}^{(2,0)}(r_0',u_0)=\frac{\partial^2}{(\partial r_0')^2}\mathring{\mathcal{F}}_{\pm}(r_0',u_0), \tag{C2}$$

where a_0 is defined in Eq. (13). Above T_c we obtain from Eq. (73)

$$\mathring{\Gamma}_{+}^{(2,0)}(r_0',u_0)=-\frac{n}{16\pi}r_0'^{-1/2}+n(n+2)\frac{u_0^2}{(4\pi)^3}r_0'^{-3/2}\times\left[\frac{n}{2}-3-4\ln\frac{3}{4}+2\ln\frac{r_0'}{(24u_0)^2}\right]+16n(n+2)\frac{u_0^3}{(4\pi)^4}r_0'^{-2}\left[\frac{\pi^2}{12}(n+8)-(n+2)\right]+O(u_0^4,u_0^4\ln u_0) \tag{C3}$$

for $r_0'>0$. Below T_c we obtain from Eq. (92)

$$\mathring{\Gamma}_{-}^{(2,0)}(r_0',u_0)=-\frac{1}{8u_0}-\frac{1}{4\pi}(-2r_0')^{-1/2}+\frac{u_0}{4\pi^2}(n+2)(-2r_0')^{-1}-\frac{u_0^2}{(4\pi)^3}(-2r_0')^{-3/2}\left[\frac{8}{3}(11n+7)-216c_1-8(n-1)c_2\right.$$
$$+\frac{\pi^2}{2}(3n^2+11n-5)-21(n-1)\left[2\mathrm{Li}_2\left(-\frac{1}{3}\right)+(\ln 3)^2\right]-8(4n+17)\ln 3+\frac{8}{3}(31n+95)\ln 2+54\,\mathrm{Li}_2\left(\frac{1}{3}\right)$$
$$\left.+16(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\right]+\frac{32(n+2)}{(-2r_0')^2}\frac{u_0^3}{(4\pi)^4}\left[6+2(n-1)\ln 3-\frac{\pi^2}{24}n(n+8)-(n+2)\ln\frac{-2r_0'}{(24u_0)^2}\right]$$
$$+O(u_0^4,u_0^4\ln u_0) \tag{C4}$$

for $r_0'<0$. The three-loop parts of Eqs. (C3) and (C4) agree with the (slightly different form of the) results of Ref. [36] [see our Eqs. (A54) and (A55)]. In order to absorb the logarithmic u_0 dependence of Eqs. (C3) and (C4), we express r_0' in terms of the correlation lengths $\xi_{\pm}$ according to Eqs. (58) and (66). This leads to the bare vertex functions up to four-loop order

$$\mathring{\Gamma}_{\pm}^{(2,0)}(r_0'(\xi_\pm,u_0),u_0)\equiv\mathring{\Gamma}_{\pm}^{(2,0)}(\xi_\pm,u_0,3)=\xi_\pm\left\{\sum_{m=0}^{4} a_{\pm m}^{(2,0)}(u_0\xi_\pm)^{m-1}+O(u_0^4\xi_\pm^4)\right\} \tag{C5}$$

with the coefficients

$$a_{+0}^{(2,0)}=0, \tag{C6}$$

$$a_{+1}^{(2,0)}=-\frac{n}{16\pi}, \tag{C7}$$

$$a_{+2}^{(2,0)}=\frac{n(n+2)}{32\pi^2}, \tag{C8}$$

$$a_{+3}^{(2,0)}=-\frac{n(n+2)}{(4\pi)^3}\left[n+\frac{160}{27}+4\ln\frac{3}{4}\right], \tag{C9}$$

$$a_{+4}^{(2,0)}=-\frac{n(n+2)}{(4\pi)^4}\left[2n^2+\frac{224}{9}n+\frac{440}{9}-\frac{4\pi^2}{9}(n+8)+\frac{16}{27}(19n-10)\ln\frac{3}{4}-\frac{64}{3}(n+8)\mathrm{Li}_2\left(-\frac{1}{3}\right)\right] \tag{C10}$$

above T_c, and

$$a_{-0}^{(2,0)}=-\frac{1}{8}, \tag{C11}$$

$$a_{-1}^{(2,0)}=-\frac{1}{4\pi}, \tag{C12}$$

$$a_{-2}^{(2,0)}=\frac{3}{8\pi^2}(n+2), \tag{C13}$$

$$a_{-3}^{(2,0)}=\frac{1}{(4\pi)^3}\left[22n^2+\frac{5938}{27}n+\frac{8420}{27}-8(n-1)c_2-21(n-1)\left[2\,\mathrm{Li}_2\left(-\frac{1}{3}\right)+(\ln 3)^2\right]-216c_1+\frac{\pi^2}{2}(3n^2+11n-5)\right.$$
$$\left.+54\,\mathrm{Li}_2\left(\frac{1}{3}\right)+\frac{8}{3}(31n+95)\ln 2-8(4n+17)\ln 3\right], \tag{C14}$$

$$a_{-4}^{(2,0)}=\frac{1}{(4\pi)^4}\left[\frac{4}{27}(567n^3+16\,481n^2+73154n+84920)-12(n-1)(n+2)\left[4c_2+42\,\mathrm{Li}_2\left(\frac{1}{3}\right)+21(\ln 3)^2\right]-1296(n+2)c_1\right.$$
$$+\frac{\pi^2}{9}(129n^3+1067n^2+2507n+1778)+\frac{4}{3}(128n^2+1523n+2534)\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{16}{27}(493n^2+2095n+2218)\ln 2$$
$$\left.-\frac{16}{27}(584n^2+1135n+868)\ln 3\right] \tag{C15}$$

below T_c. The coefficients up to three-loop order ($m=3$) agree with the (slightly different form of the) coefficients of Ref. [36] [see our Eqs. (A54) and (A55)]. For the special case $n=1$, Eqs. (C11)–(C14) agree with the numerical values of $a_{-m}^{(\Gamma)}$ in Table 2 of Ref. [33].

Substituting Eqs. (C5)–(C15) into the right-hand side of Eq. (112) for $d=3$, and using Eq. (116) at $\xi_\pm=\mu^{-1}$ for $d=3$ we obtain the power series for $F_\pm(1,u,3)$, Eq. (145), up to four-loop order. The resulting coefficients $c_{Fm}^{\pm}$ agree with those given in Eqs. (146)–(155).

APPENDIX D: AMPLITUDE FUNCTIONS $f_{\pm}^{(0,0)}$ AND $f_{\pm}^{(1,0)}$

Equations (144), (95)–(105) , and (138) lead to

$$f_{+}^{(0,0)}(1,u,3)=-\frac{1}{3}n-n(n+2)u-\frac{4}{27}n(n+2)\left(27n+110+108\ln\frac{4}{3}\right)u^2-\frac{8}{27}n(n+2)\Big[54n^2+215n-938+(604n+4184)\ln\frac{4}{3}-24(n+8)\Big[\pi^2+12\mathrm{Li}_2\left(-\frac{1}{3}\right)\Big]-9(n+8)\pi^2\ln(96\pi u)\Big]u^3+O(u^4)-\frac{1}{8}Q_{+}(1,u,3)^2A(u,1), \tag{D1}$$

$$f_{+}^{(1,0)}(1,u,3)=n+4n(n+2)u+n\left(16n^2+\frac{2492}{27}n+\frac{3256}{27}+16(n+2)\ln\frac{4}{3}\right)u^2+\frac{8}{27}n\Big[216n^3+2008n^2+6826n+7348+8(n+2)(89n+550)\ln\frac{4}{3}-(n+2)(n+8)\Big[15\pi^2+288\mathrm{Li}_2\left(-\frac{1}{3}\right)\Big]\Big]u^3+O(u^4)+\frac{1}{2}Q_{+}(1,u,3)A(u,1), \tag{D2}$$

$$\begin{aligned}f_{-}^{(0,0)}(1,u,3)=&-\frac{1}{64u}-\frac{n}{16}-\frac{1}{12}-\frac{1}{4}\left(n^2-\frac{247}{27}n+\frac{370}{27}+8(n-1)\ln 3\right)u-\frac{1}{4}\Big[4n^3-\frac{1909}{54}n^2-\frac{2309}{9}n+\frac{7676}{27}+\Big(\frac{2416}{27}n^2\\&+\frac{14\,656}{27}n+\frac{11\,872}{27}\Big)\ln 3-\left(\frac{1376}{27}n^2+\frac{17\,504}{27}n+\frac{39\,008}{27}\right)\ln 2-(80n+352)\zeta(3)+84(n-1)\\&\times\Big[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\Big]-\frac{1}{9}(22n^2-122n-602)\pi^2+\frac{1}{3}(128n^2+1280n+1400)\mathrm{Li}_2\left(-\frac{1}{3}\right)+864c_1\\&+32(n-1)c_2\Big]u^2-\Big\{4n^4+\Big[\frac{3968}{27}\ln 3-\frac{5035}{144}-\frac{44}{9}\pi^2+\frac{256}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)-\frac{2752}{27}\ln 2\Big]n^3+\Big[1536\mathrm{Li}_2\left(-\frac{1}{3}\right)\\&-12\pi^2-\frac{1856}{27}-2112\ln 2+\frac{5680}{3}\ln 3+168\Big[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\Big]-\frac{\pi^4}{3}+86\zeta(3)+80\zeta(5)+64c_2\Big]n^2\\&+\Big[\frac{23\,4173}{54}+\frac{1052}{3}\pi^2-\frac{11\,9360}{9}\ln 2+\frac{67\,504}{9}\ln 3+7760\mathrm{Li}_2\left(-\frac{1}{3}\right)+1032\zeta(3)+1176\Big[(\ln 3)^2+2\,\mathrm{Li}_2\left(\frac{1}{3}\right)\Big]\\&+1728c_1+448c_2\\&-\frac{62}{15}\pi^4+2200\zeta(5)\Big]n+\frac{72\,7721}{54}+\frac{26\,3392}{27}\ln 3-\frac{62\,4128}{27}\ln 2+\frac{9632}{9}\pi^2+\frac{22\,400}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)\\&-1344\Big[(\ln 3)^2+2\mathrm{Li}_2\left(\frac{1}{3}\right)\Big]-\frac{176}{15}\pi^4+13\,824c_1-512c_2+3256\zeta(3)+7440\zeta(5)-\frac{8}{3}n(n+2)\\&(n+8)\pi^2\ln(4\pi u)-(4\pi)^4a_{-4}^{(\Gamma)}\Big\}u^3+O(u^4)-\frac{1}{32}Q_{-}(1,u,3)^2A(u,1)\,,\end{aligned} \tag{D3}$$

with $a_{-4}^{(\Gamma)}$ given in Eq. (105), and

$$\begin{aligned}f_{-}^{(1,0)}(1,u,3)=&-\frac{1}{8u}-\frac{n}{2}-\left(2n^2-\frac{247}{27}n+\frac{586}{27}+8(n-1)\ln 3\right)u+\frac{1}{2}\Big[-16n^3+\Big(\frac{1810}{27}-\frac{256}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)+\frac{2752}{27}\ln 2-\frac{4832}{27}\ln 3\\&-\frac{10}{9}\pi^2\Big)n^2+\Big(\frac{13\,772}{27}-\frac{25856}{27}\ln 3+\frac{26\,080}{27}\ln 2-\frac{442}{9}\pi^2+320\zeta(3)-\frac{2560}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)-84\Big[2\mathrm{Li}_2\left(\frac{1}{3}\right)+(\ln 3)^2\Big]\\&-32c_2\Big)n-240+\frac{50565}{27}\ln 2-\frac{9056}{27}\ln 3-\frac{1114}{9}\pi^2+84\Big[(\ln 3)^2+2\mathrm{Li}_2\left(\frac{1}{3}\right)\Big]-\frac{3448}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)+1408\zeta(3)\\&-864c_1+32c_2\Big]u^2+\frac{1}{2}\Big[\Big(\frac{11\,008}{9}\ln 2-\frac{12\,416}{9}\ln 3-\frac{8009}{18}-36\pi^2-1024\mathrm{Li}_2\left(-\frac{1}{3}\right)+24\zeta(3)\Big)n^3-64n^4\end{aligned}$$

$$+\left(\frac{515\,264}{27}\ln 2-\frac{401\,344}{27}\ln 3-\frac{392\,458}{27}-\frac{8108}{9}\pi^2-320c_2-840\left[(\ln 3)^2+2\mathrm{Li}_2\left(\frac{1}{3}\right)\right]\frac{51\,200}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)\right.$$
$$\left.-928\zeta(3)+8\pi^4-1280\zeta(5)\right)n^2+\left(\frac{2\,532\,416}{27}\ln 2-\frac{1\,465\,216}{27}\ln 3-\frac{2\,998\,732}{27}-\frac{247472}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)\right.$$
$$\left.-\frac{52460}{9}\pi^2-13\,760\zeta(3)-35\,200\zeta(5)-4872\left[(\ln 3)^2+2\mathrm{Li}_2\left(\frac{1}{3}\right)\right]-8640c_1-1856c_2+\frac{416}{5}\pi^4\right)n-\frac{5\,481\,052}{27}$$
$$+\frac{3\,724\,160}{27}\ln 2-\frac{1\,498\,624}{27}\ln 3-\frac{283\,616}{3}\mathrm{Li}_2\left(-\frac{1}{3}\right)+5712\left[(\ln 3)^2+2\mathrm{Li}_2\left(\frac{1}{3}\right)\right]-\frac{88\,040}{9}\pi^2-48\,192\zeta(3)$$
$$\left.+\frac{1056}{5}\pi^4-119\,040\zeta(5)-58\,752c_1+2176c_2+\frac{1}{2}(4\pi)^4a_{r4}\right]u^3+O(u^4)-\frac{1}{4}Q_-(1,u,3)A(u,1). \tag{D4}$$

It is understood that in the last terms $\sim Q_\pm^2 A$ and $\sim Q_\pm A$ of Eqs. (D1)–(D4), the functions $Q_\pm(1,u,3)$ and $A(u,1)$ should be inserted in their *perturbative* four-loop form.

We see that $f_\pm^{(0,0)}$ and $f_\pm^{(1,0)}$ contain logarithmic terms due to the logarithmic terms of the functions $Q_\pm(1,u,3)$ as given by Eqs. (B8), (124), and (131). Furthermore, $f_+^{(0,0)}$ and $f_-^{(0,0)}$ contain logarithmic terms $\sim u^3\ln u$ arising from those parts of the four-loop diagrams of type B in Fig. 1 that yield the $d=3$ pole term, Eq. (28). Since such terms are not temperature dependent they are not present in $f_\pm^{(1,0)}$, therefore the functions $f_\pm^{(1,0)}(1,u,3)-\frac{1}{2}q_\pm(1,u,3)A(u,1)$ are free of logarithms, as seen explicitly in Eqs. (D2) and (D4).

APPENDIX E: ASYMPTOTIC CRITICAL BEHAVIOR

In order to evaluate the expression for the specific heat, Eqs. (134) and (135), asymptotically ($\mu l_\pm=\xi_\pm^{-1}\to 0$), we need to determine the asymptotic behavior of

$$\tilde{A}(u(l_\pm),u)\equiv A(u(l_\pm),\varepsilon)\int_u^{u(l_\pm)}\frac{2\zeta_r(u')-\varepsilon}{\beta_u(u',\varepsilon)}du' \tag{E1}$$

for $u(l_\pm)\to u^\star$. Substituting the integral representation [31]

$$A(u,\varepsilon)=4\int_0^u du'\frac{B(u')}{\beta_u(u',\varepsilon)}\exp\left(\int_u^{u'}du''\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}\right) \tag{E2}$$

into Eq. (E1), we obtain

$$\tilde{A}(u(l_\pm),u)=4\int_0^{u(l_\pm)}du'\frac{B(u')}{\beta_u(u',\varepsilon)}\times\exp\left(\int_u^{u'}du''\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}\right). \tag{E3}$$

In the following it is necessary to distinguish the cases $2\zeta_r(u^\star)-\varepsilon>0$ and $2\zeta_r(u^\star)-\varepsilon<0$ corresponding to the cases $\alpha<0$ and $\alpha>0$, respectively.

For $\alpha<0$, the quantity $\tilde{A}$ has a finite limit,

$$\tilde{A}(u^\star,u)=4\int_0^{u^\star}du'\frac{B(u')}{\beta_u(u',\varepsilon)}\exp\left(\int_u^{u'}du''\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}\right). \tag{E4}$$

Using

$$2\zeta_r(u^\star)-\varepsilon=-\frac{\alpha}{\nu}, \tag{E5}$$

we obtain from Eq. (E3) the leading critical behavior, apart from Wegner corrections,

$$\tilde{A}(u(l_\pm),u)=\tilde{A}(u^\star,u)-4\frac{\nu}{\alpha}B(u^\star)\tilde{C}^{-2}l_\pm^{-\alpha/\nu}+O(l_\pm^{\omega-\alpha/\nu}), \tag{E6}$$

where $\tilde{C}$ is given by Eq. (180). This leads to the expression for $B_{cr}^+=B_{cr}^-$, Eq. (189), for $\alpha<0$.

For $\alpha>0$, the quantity $\tilde{A}$ diverges for $u(l_\pm)\to u^\star$. In order to extract the divergent part, we rewrite Eq. (E3) as

$$\tilde{A}(u,(l_\pm),u)=4\int_0^{u(l_\pm)}\tilde{f}(u',u)\tilde{G}(u',u)du', \tag{E7}$$

with

$$\tilde{f}(u',u)=B(u')\exp\left(\int_u^{u'}\frac{2\zeta_r(u')-2\zeta_r(u^\star)}{\beta_u(u'',\varepsilon)}du''\right), \tag{E8}$$

$$\tilde{G}(u',u)=\frac{1}{\beta_u(u',\varepsilon)}\exp\left(\int_u^{u'}\left[-\frac{\alpha}{\nu\beta_u(u'',\varepsilon)}\right]du''\right) \tag{E9}$$

$$=\frac{\partial}{\partial u'}\tilde{g}(u',u), \tag{E10}$$

$$\tilde{g}(u',u)=-\frac{\nu}{\alpha}\exp\left(\int_u^{u'}\left[-\frac{\alpha}{\nu\beta_u(u'',\varepsilon)}\right]du''\right). \tag{E11}$$

Partial integration of Eq. (E7) yields

$$\tilde{A}(u(l_\pm),u) = -4\frac{\nu}{\alpha}B(u(l_\pm))\exp\left(\int_u^{u(l_\pm)}\frac{2\zeta_r(u')-\varepsilon}{\beta_u(u',\varepsilon)}du'\right) - 4\int_0^{u(l_\pm)}\left[\frac{\partial}{\partial u'}\tilde{f}(u',u)\right]\tilde{g}(u',u)du', \quad \text{(E12)}$$

where we have used

$$\lim_{u'\to 0}\tilde{f}(u',u)\,\tilde{g}(u',u)=0. \quad \text{(E13)}$$

The latter property follows from the fact that $\tilde{f}(u',u)\tilde{g}(u',u)$ is proportional to

$$\exp\left(\int_u^{u'}\frac{2\zeta_r(u')-\varepsilon}{\beta_u(u',\varepsilon)}du'\right), \quad \text{(E14)}$$

which vanishes for $u'\to 0$ because of $\zeta_r(0)=0$ and $\beta_u(u',\varepsilon)=-\varepsilon u'+O(u'^2)$. From Eq. (E12) we obtain asymptotically for $\alpha>0$,

$$\tilde{A}(u(l_\pm),u) = -4\frac{\nu}{\alpha}B(u^\star)\tilde{C}^{-2}l_\pm^{-\alpha/\nu}+\tilde{B}+O(l_\pm^{\omega-\alpha/\nu}) \quad \text{(E15)}$$

with the finite constant

$$\tilde{B} = -4\frac{\nu}{\alpha}\int_0^{u^\star}\left(\frac{2B(u')[\zeta_r(u')-\zeta_r^\star]}{\beta_u(u',\varepsilon)}+\frac{\partial B(u')}{\partial u'}\right)\times\left[\exp\left(\int_u^{u'}\frac{2\zeta_r(u'')-\varepsilon}{\beta_u(u'',\varepsilon)}du''\right)\right]du'. \quad \text{(E16)}$$

This leads to the expression for $B_{cr}^+=B_{cr}^-$, Eq. (190), for $\alpha>0$.

In order to derive the singular part of the free energy, Eq. (183), we formally split the integral in the second term of Eq. (133) as follows

$$\xi_\pm^{-d}\int_1^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'} = \xi_\pm^{-d}\int_1^{0}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'} + \xi_\pm^{-d}\int_0^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}. \quad \text{(E17)}$$

The first term on the right-hand side of Eq. (E17) yields

$$\xi_\pm^{-d}l_\pm^{\varepsilon}\left[\exp\left(\int_{l_\pm}^1 2\zeta_r\frac{d\tilde{l}}{\tilde{l}}\right)\right]\left\{\int_1^0 B(u(l'))\times\left[\exp\left(\int_1^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\} = \mu^{-\varepsilon}q_\pm(1,u(l_\pm),d)^{-2}r^2\times\left\{\int_1^0 B(u(l'))\times\left[\exp\int_1^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right]\frac{dl'}{l'}\right\}, \quad \text{(E18)}$$

where we have used Eq. (176); thus this term yields only a regular contribution $\sim t^2$ to the free energy. Substituting Eqs. (E17) and (E18) into Eq.(133) we obtain Eq. (136), with the singular part

$$f_s^\pm(t)=A_d\xi_\pm^{-d}\left\{f_\pm^{(0,0)}(1,u(l_\pm),d)+\frac{1}{2}q_\pm(1,u(l_\pm),d)^2\times\int_0^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\}. \quad \text{(E19)}$$

For $l_\pm\to 0$ the integral in Eq. (E19) yields

$$\lim_{l_\pm\to 0}\int_0^{l_\pm}B(u(l'))\left[\exp\left(\int_{l_\pm}^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'} = -B(u^\star)\frac{\nu}{\alpha}. \quad \text{(E20)}$$

This leads to the asymptotic expression for $f_s^\pm(t)$, Eq. (183). For an appropriate representation of the regular contribution,

$$-\frac{1}{2}B_{cr}t^2=\frac{1}{8}A_d\mu^{-\varepsilon}r^2\left\{A(u,\varepsilon)-4\int_0^1 B(u(l'))\times\left[\exp\left(\int_1^{l'}(2\zeta_r-\varepsilon)\frac{dl''}{l''}\right)\right]\frac{dl'}{l'}\right\}, \quad \text{(E21)}$$

it is necessary to distinguish the cases $\alpha<0$ and $\alpha>0$, see Eqs. (E1)–(E16) above.

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